CSE 190 / Math 152 - Introduction to Quantum Computing
Homework 2
Due **Tuesday, April 16th, 1:30pm**

Instructions: There may be opportunities to work in groups for future assignments, but since this assignment is the basis for all future work in this class, it is important that it is completed individually.

It is highly recommended (though not required) that you type your answers. It is your responsibility to make any handwriting clear and legible for grading. A LaTeX template for the homework is provided on Canvas.

For many of the problems below, I ask you to "prove" some fact. In general in this class, there is no specific structure of a proof that I am looking for. Most of the proofs of the identities below can be shown by just computing two quantities and showing that they are evidently the same.

We will only be grading some of the problems below for correctness. However, because all of the concepts are important, we will not reveal which problems are being graded for correctness until after the assignment has been submitted. The remaining problems will be graded for completeness (i.e., does it look like there was a good-faith effort to solve the problem?).

Problems:

1. **Dirac notation and tensor products**

   Recall that we use $|0\rangle$ to denote the vector $\binom{1}{0}$, and $|1\rangle$ to denote the vector $\binom{0}{1}$. Therefore, we can express every qubit as

   $$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$$

   where $\alpha, \beta \in \mathbb{C}$ such that $|\alpha|^2 + |\beta|^2 = 1$. In other words, $|\psi\rangle$ is the vector $\binom{\alpha}{\beta}$. We have the following basic single-qubit unitary matrices:

   $$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

   (a) Let $|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$ be any single-qubit quantum state. Write the following states as a linear combination of $|0\rangle$ and $|1\rangle$: $H|\psi\rangle$, $X|\psi\rangle$, and $HZH|\psi\rangle$. Based on these computations, what identity between single-qubit operators can you prove?

   For any two qubits $|\psi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle$ and $|\varphi\rangle = \beta_0 |0\rangle + \beta_1 |1\rangle$, we define their *tensor product* $|\psi\rangle \otimes |\varphi\rangle$ as the natural state that is a combination of both qubits:

   $$\begin{aligned} |\psi\rangle \otimes |\varphi\rangle &= \alpha_0\beta_0 |0\rangle \otimes |0\rangle + \alpha_0\beta_1 |0\rangle \otimes |1\rangle + \alpha_1\beta_0 |1\rangle \otimes |0\rangle + \alpha_1\beta_1 |1\rangle \otimes |1\rangle \\ &= \alpha_0\beta_0 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \alpha_0\beta_1 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \alpha_1\beta_0 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \alpha_1\beta_1 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha_0\beta_0 \\ \alpha_0\beta_1 \\ \alpha_1\beta_0 \\ \alpha_1\beta_1 \end{pmatrix} \end{aligned}$$

   where for each $x, y \in \{0, 1\}$, we've defined $|x\rangle \otimes |y\rangle$ to be the length-4 vector with a 1 in position $(xy)_2$ (i.e., when the number is written in binary). We say that a two-qubit state is *entangled* if it cannot be written as the tensor product of two single-qubit states.

   (b) Give an example of a 2-qubit operation U such that $U(|0\rangle \otimes |0\rangle)$ is entangled. (Make sure to check that U is unitary!)

   Given two single-qubit unitaries U and V, we can define their tensor product to act on two separate subsystems. In particular, we define their tensor product $U \otimes V$ to be the unique 4×4 matrix such that

   $$(U \otimes V)(|\psi\rangle \otimes |\varphi\rangle) = (U|\psi\rangle) \otimes (V|\varphi\rangle) \tag{1}$$

   for all single-qubit states $|\psi\rangle$ and $|\varphi\rangle$.

   (c) Let $U = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$ and $V = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}$ be arbitrary 2×2 unitary matrices. Write out the tensor product matrix $U \otimes V$.

   (d) Using the characterization in Equation 1, prove that what you wrote down in the previous is correct. *Hint: A matrix is uniquely determined by its action on a set of basis states. A basis for two-qubit states is* $\{|0\rangle \otimes |0\rangle, |0\rangle \otimes |1\rangle, |1\rangle \otimes |0\rangle, |1\rangle \otimes |1\rangle\}$.

2. **Inner products and bases**

Let $|\psi\rangle$ be an arbitrary quantum state. In other words, we can write

$$|\psi\rangle = \sum_{x\in\{0,1\}^n} \alpha_x |x\rangle$$

with $\sum_{x\in\{0,1\}^n} |\alpha_x|^2 = 1$. Recall, $|\psi\rangle$ is a column vector. We define the congugate transpose of $|\psi\rangle$ as the state (now represented by a row vector)

$$\langle\psi| = \sum_{x\in\{0,1\}^n} \alpha_x^* \langle x| .$$

We define the *inner product* of two quantum states $|\psi\rangle = \sum_{x\in\{0,1\}^n} \alpha_x |x\rangle$ and $|\varphi\rangle = \sum_{x\in\{0,1\}^n} \beta_x |x\rangle$ as

$$\langle\psi| |\varphi\rangle := \langle\psi|\varphi\rangle = \sum_{x\in\{0,1\}^n} \alpha_x^* \beta_x$$

(a) Let $|\psi_1\rangle = \frac{|0\rangle + i|1\rangle}{\sqrt{2}}$, $|\psi_2\rangle = \frac{|0\rangle - i|1\rangle}{\sqrt{2}}$, and $|\psi_3\rangle = \frac{|0\rangle + \sqrt{2}|1\rangle}{\sqrt{3}}$. Compute the following inner products: $\langle\psi_1|\psi_2\rangle, \langle\psi_1|\psi_3\rangle, \langle\psi_2|\psi_3\rangle$.

(b) Prove that for any n-qubit state $|\psi\rangle = \sum_{x\in\{0,1\}^n} \alpha_x |x\rangle$, we have $\langle\psi|\psi\rangle = 1$.

(c) Let $|\psi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle$ and $|\varphi\rangle = \beta_0 |0\rangle + \beta_1 |1\rangle$ be arbitrary quantum states such that $\langle\psi|\varphi\rangle = 0$. Give the unitary U such that $U|\psi\rangle = |0\rangle$ and $U|\varphi\rangle = |1\rangle$. Conclude that we can use the unitary U to determine whether or not some unknown state is $|\psi\rangle$ or $|\varphi\rangle$.

An orthonormal basis for n-qubit quantum state vectors is a set of 2^n quantum states $\{|\psi_i\rangle\}_{i=1}^{2^n}$ with the following property: $\langle\psi_i|\psi_j\rangle = 0$ for all $i \neq j$. As a standard linear-algebraic consequence, we can write *every* n-qubit state $|\varphi\rangle$ as a linear combination of those states:

$$|\varphi\rangle = \sum_{i=1}^{2^n} \alpha_i |\psi_i\rangle$$

for some complex amplitudes $\alpha_i \in \mathbb{C}$. The *computational basis* is the orthonormal basis consisting of the basis vectors $\{|x\rangle\}_{x\in\{0,1\}^n}$. For the following problems, let $\{|\psi_i\rangle\}_{i=1}^{2^n}$ be an arbitary orthonormal basis for n-qubit quantum states.

(d) Show that the basis expansion, $|\varphi\rangle = \sum_{i=1}^{2^n} \alpha_i |\psi_i\rangle$, is unique. Specifically, imagine some other expansion $|\varphi\rangle = \sum_{i=1}^{2^n} \beta_i |\psi_i\rangle$, and show that $\alpha_i = \beta_i$ for all i. *Hint: use inner products.*

(e) Given that $|\varphi\rangle = \sum_{i=1}^{2^n} \alpha_i |\psi_i\rangle$, show that $\sum_{i=1}^{2^n} |\alpha_i|^2 = 1$. In other words, the sum of the amplitude magnitudes squared equals 1 regardless of which basis in which we write $|\varphi\rangle$. We have only defined this property in the computational basis.

(f) Show that for any n-qubit unitary U, $\{U|\psi_i\rangle\}_{i=1}^{2^n}$ is also an orthonormal basis.

(g) Let $|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$ and $|-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$. Write the state $\frac{|0\rangle + i|1\rangle}{\sqrt{2}}$ in the $\{|+\rangle, |-\rangle\}$ basis.