

CSE 190 / Math 152 - Introduction to Quantum Computing
Homework 3

Due **Tuesday, April 23rd, 1:30pm**

Instructions: **You may work individually or in a team of 2 people.** You may switch teams for different assignments. Please ensure your name(s) and PID(s) are clearly visible on the first page of your submission, and then upload the PDF to Gradescope. If working in a group, submit only one submission per group: one partner uploads the submission through their Gradescope account and then adds the other group member to the Gradescope submission by selecting their name in the “Add Group Members” dialog box. You will need to re-add your group member every time you resubmit a new version of your assignment.

It is highly recommended (though not required) that you type your answers. It is your responsibility to make any handwriting clear and legible for grading. A LaTeX template for the homework is provided on Canvas.

We will only be grading some of the problems below for correctness. However, because all of the concepts are important, we will not reveal which problems are being graded for correctness until after the assignment has been submitted. The remaining problems will be graded for completeness (i.e., does it look like there was a good-faith effort to solve the problem?).

Problems:

1. No-Cloning Theorem with junk

In class, we proved the following theorem: There is no two-qubit unitary U for which

$$U(|\psi\rangle \otimes |0\rangle) = |\psi\rangle \otimes |\psi\rangle$$

for all single-qubit states $|\psi\rangle$. However, this is not the only possible definition of a cloning unitary. For example, we might have a 3-qubit cloning unitary U such that

$$U(|\psi\rangle \otimes |0\rangle \otimes |0\rangle) = |\psi\rangle \otimes |\psi\rangle \otimes |\text{junk}(\psi)\rangle$$

for all single-qubit states $|\psi\rangle$ and some arbitrary state $|\text{junk}(\psi)\rangle$ that depends on the cloned state $|\psi\rangle$. In other words, in this setting, the unitary copies the first qubit into the second qubit, but also produces some arbitrary state in the third qubit. One can check that the exact proof outline we saw in class for the original No-Cloning Theorem no longer works, so let's try something else.

Suppose there exists such a cloner with junk U . Recall that $|+\rangle := \frac{|0\rangle+|1\rangle}{\sqrt{2}}$. We can use the definition of the cloner U to show that

$$U(|+\rangle \otimes |0\rangle \otimes |0\rangle) = |+\rangle \otimes |+\rangle \otimes |\text{junk}(+)\rangle. \quad (1)$$

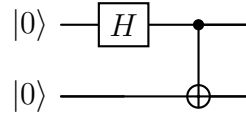
Alternatively, we can write $U(|+\rangle \otimes |0\rangle \otimes |0\rangle)$ using the linearity of matrix-vector multiplication as

$$U\left(\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) \otimes |00\rangle\right) = U\left(\frac{|000\rangle + |100\rangle}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}(U|000\rangle + U|100\rangle) \quad (2)$$

Using the definition of the cloner U on states $|000\rangle$ and $|100\rangle$, show that the state derived in Equation (1) is different from the state in Equation (2). Conclude that cloning states with junk is impossible. (Note: You need to prove that the two states are different.)

2. Circuit building

In this problem, you will practice building a quantum circuit for a quantum state. For example, the circuit which constructs the state $\frac{|00\rangle+|11\rangle}{\sqrt{2}}$ is the following:



Some of you may find it useful to play around with quantum circuits in the Qiskit SDK. Information on how to install Qiskit is [here](#) with tutorials [here](#). To do simulations of quantum circuits on your classical computer, you will also need the [Aer](#) library, which can be installed using the [Python package manager](#) using the command `pip install qiskit-aer`. The Python code below prints the state vector associated with the circuit above:

```
from qiskit_aer import Aer
from qiskit import QuantumCircuit

qc = QuantumCircuit(2)
qc.h(0)
qc.cx(0,1)

backend = Aer.get_backend("statevector_simulator")
print(backend.run(qc).result().get_statevector())
```

For each of the following quantum states, draw a quantum circuit which constructs it. Assume that all qubits are initialized to $|0\rangle$. The circuit above was drawn in LaTeX using the [quantikz package](#), but it is also fine to draw the circuit by hand. You may use the following single-qubit gates:

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \quad RY = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & -\sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix}$$

In Qiskit, the final gate is `RYGate(1.91063)` where the angle of rotation comes from the fact that $2 \arcsin(\sqrt{2}/3) \approx 1.91063$. You are also allowed to use the singly or doubly controlled versions of the gates above. Recall that controlled- X is the CNOT gate and controlled-controlled- X is the Toffoli gate.

- $\frac{|000\rangle+|111\rangle}{\sqrt{2}}$
- $\frac{|00\rangle+i|01\rangle+|10\rangle+i|11\rangle}{2}$
- $\frac{|011\rangle+|101\rangle+|110\rangle}{\sqrt{3}}$.

3. The Clifford circuits are not universal

The quantum operations that can be built using circuits of Hadamard (H), Phase (S), and CNOT gates are called the *Clifford operations*. While the Clifford operations are extremely important in quantum computation (we will see many uses in this class), it turns out they they also are very limited. We say that a set of gates is *exactly universal* for a set of unitary operations if for every unitary in the set, there is a quantum circuit which constructs it up to a complex scalar factor of magnitude 1.

For example, Clifford circuits are exactly universal for the set containing the single unitary $\frac{1}{\sqrt{2}} \begin{pmatrix} -1 & -1 \\ -i & i \end{pmatrix}$ since the circuit



constructs the unitary up to a -1 scalar factor—that is, $SH = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$.

- (a) Show that Clifford circuits are exactly universal for the Pauli operators:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Hint: Multiply two of the matrices above. What do you get? Also, use the identity that you proved in the previous homework: $HZH = X$.

- (b) Show that the Clifford circuits are *not* exactly universal for the set of all two-qubit unitary operations. Correctness will be judged based on whether or not you have the right idea even if your proof isn't entirely formal.

Hint: Show that each entry of the unitary matrix of a Clifford circuit has a special form. Since the entries of an arbitrary unitary matrix do not have that form, reach the conclusion.