CSE 190 / Math 152 - Introduction to Quantum Computing
Homework 3
Due **Thursday, April 24th, 11:59pm**

Instructions: It is highly recommended (though not required) that you type your answers. It is your responsibility to make any handwriting clear and legible for grading. A LaTeX template for the homework is provided on Canvas. For ease of grading, please start each new problem on a separate page.

We will only be grading some of the problems below for correctness. However, because all of the concepts are important, we will not reveal which problems are being graded for correctness until after the assignment has been submitted. The remaining problems will be graded for completeness (i.e., does it look like there was a good-faith effort to solve the problem?).

Problems:

1. **Concept check**

   Complete the assignment "Homework 3 - Concept Check" on Gradescope.

2. **No-Cloning Theorem with junk**

   In class, we proved the following theorem: there is no two-qubit unitary U for which

   $$U(|\psi\rangle \otimes |0\rangle) = |\psi\rangle \otimes |\psi\rangle$$

   for all single-qubit states $|\psi\rangle$. However, this is not the only possible definition of a cloning unitary. For example, we might have a 3-qubit cloning unitary U such that

   $$U(|\psi\rangle \otimes |0\rangle \otimes |0\rangle) = |\psi\rangle \otimes |\psi\rangle \otimes |\text{junk}(\psi)\rangle$$

   for all single-qubit states $|\psi\rangle$ and some arbitrary state $|\text{junk}(\psi)\rangle$ that depends on the cloned state $|\psi\rangle$. In other words, in this setting, the unitary copies the first qubit into the second qubit, but also produces some arbitrary state in the third qubit. One can check that the exact proof outline we saw in class for the original No-Cloning Theorem no longer works, so let's try something else.

   Suppose there exists such a cloner U. Recall that $|+\rangle := \frac{|0\rangle+|1\rangle}{\sqrt{2}}$. We can use the definition of the cloner U to show that

   $$U(|+\rangle \otimes |0\rangle \otimes |0\rangle) = |+\rangle \otimes |+\rangle \otimes |\text{junk}(+)\rangle \,. \tag{1}$$

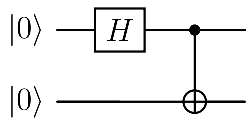
   Alternatively, we can write $U(|+\rangle \otimes |0\rangle \otimes |0\rangle)$ using the linearity of matrix-vector multiplication as

   $$U\left(\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) \otimes |00\rangle\right) = U\left(\frac{|000\rangle + |100\rangle}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}\left(U|000\rangle + U|100\rangle\right) \tag{2}$$

   Using the definition of the cloner U on states $|000\rangle$ and $|100\rangle$, show that the state derived in Equation (1) is different from the state in Equation (2). Conclude that cloning states with junk is impossible. (Note: You need to prove that the two states are different.)

3. **Circuit building**

In this problem, you will practice building a quantum circuit for a quantum state. For example, the circuit which constructs the state $\frac{|00\rangle+|11\rangle}{\sqrt{2}}$ is the following:



Some of you may find it useful to play around with quantum circuits in the Qiskit SDK. Information on how to install Qiskit is here with tutorials here. The Python code below prints the state vector associated with the circuit above:

```
from qiskit_aer import Aer
from qiskit import QuantumCircuit

qc = QuantumCircuit(2)
qc.h(0)
qc.cx(0,1)

backend = Aer.get_backend("statevector_simulator")
print(backend.run(qc).result().get_statevector())
```

For each of the following quantum states, draw a quantum circuit which constructs it. Assume that all qubits are initialized to $|0\rangle$. The circuit above was drawn in LaTeX using the quantikz package, but it is also fine to draw the circuit by hand. You may use the following single-qubit gates:

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}, \quad X = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad S = \begin{pmatrix}1 & 0\\ 0 & i\end{pmatrix}, \quad RY = \frac{1}{\sqrt{3}}\begin{pmatrix}1 & -\sqrt{2}\\ \sqrt{2} & 1\end{pmatrix}$$

In Qiskit, the final gate is `RYGate(1.91063)` where the angle of rotation comes from the fact that $2\arcsin(\sqrt{2/3}) \approx 1.91063$. You are also allowed to use the singly or doubly controlled versions of the gates above. Recall that controlled-X is the CNOT gate and controlled-controlled-X is the Toffoli gate.

(a) $\frac{|000\rangle+|111\rangle}{\sqrt{2}}$

(b) $\frac{|00\rangle+i|01\rangle+|10\rangle+i|11\rangle}{2}$

(c) $\frac{|011\rangle+|101\rangle+|110\rangle}{\sqrt{3}}$.