Math 154 - Discrete Math and Graph Theory
Homework 2
Due **Tuesday, January 21st, 11:59pm**

Instructions: There may be opportunities to work in groups for future assignments, but since these initial assignments are the basis for all future work in this class, it is important that it is completed individually.

It is highly recommended (though not required) that you type your answers. It is your responsibility to make any handwriting clear and legible for grading. A LaTeX template for the homework is provided on Canvas. Please start each problem on a new page.

We will only be grading some of the problems below for correctness. However, because all of the concepts are important, we will not reveal which problems are being graded for correctness until after the assignment has been submitted. The remaining problems will be graded for completeness (i.e., does it look like there was a good-faith effort to solve the problem?).

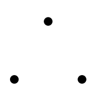
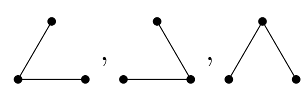
Additional textbook questions for practice (not graded): 2.7, 2.8

Problems:

1. **Counting connected component configurations**

   Let G be a simple undirected graph with k connected components, each of size n. That is, G has kn total vertices. How many ways are there to add $k-1$ edges to G so that it becomes connected?

   For example, when $n = 1$ and $k = 3$, the initial graph G must be three unconnected vertices. There are 3 ways to make the graph connected by adding 2 edges:

   $G =$  Ways to connect: 

   In general, when $n = 1$, the number of ways to form a connected graph is given by k^{k-2}. Your goal is to find a solution for general n and k.

2. **Why walk when you could just teleport?**

   Let $G = (V, E)$ be any simple undirected graph on n vertices representing a number of locations (the vertices) connected by rickety bridges (the edges). Consider a traversal of these bridges which consists of a sequence of vertices $v_1, v_2, \ldots, v_m$ (the sequence can have repetitions). Let's suppose that whenever a bridge is crossed, it collapses and cannot be used again.

   We say that the ith step of the traversal is a *teleport* if

   - That edge never existed in G: $\{v_i, v_{i+1}\} \notin E$; or
   - That edge was already crossed: Exists $j < i$ such that $\{v_i, v_{i+1}\} = \{v_j, v_{j+1}\}$.

   Notice that if there are no teleports, then the traversal is our usual definition of a walk.

   Show that for any simple connected undirected graph $G = (V, E)$, there is a traversal $v_1, \ldots, v_m$ of the vertices such that the following two conditions hold:

- Every edge of G is appears at least once: that is, for all $\{x, y\} \in E$ there exists an $i \in \{1, \ldots, m-1\}$ such that $\{x, y\} = \{v_i, v_{i+1}\}$.
- Every vertex of G is involved in at most one teleport: that is, for all $x \in V$, there is at most one $i \in \{1, \ldots, m-1\}$ such that the ith step was a teleport and $x \in \{v_i, v_{i+1}\}$.

3. **A curious graph**

   Prove/disprove: for every integer n, there is a simple undirected graph on n vertices satisfying the following conditions:

   - Has a Hamiltonian path
   - Does not have a Hamiltonian cycle
   - Has an Eulerian trail
   - Does not have an Eulerian tour

4. **Round robin tournaments**

   A round-robin tournament is a competition format in which every competitor plays one game against each of the other competitors. Let's assume that the game between two competitors always has a winner and loser. Prove that after all games have been played, it is possible arrange the players in a line such that the first player won the game against the second player, the second player won the game against the third player, and so on.

   Hint: Notice that in a round-robin tournament, every subset of the competitors constitutes its own round-robin tournament with a smaller number of players. Therefore, for a given round-robin tournament, we can use induction on smaller groups of players.