Math 154 - Discrete Math and Graph Theory
Homework 3
Due **Monday, February 3rd, 11:59pm**

Instructions: There may be opportunities to work in groups for future assignments, but since these initial assignments are the basis for all future work in this class, it is important that it is completed individually.

It is highly recommended (though not required) that you type your answers. It is your responsibility to make any handwriting clear and legible for grading. A LaTeX template for the homework is provided on Canvas. Please start each problem on a new page.

We will only be grading some of the problems below for correctness. However, because all of the concepts are important, we will not reveal which problems are being graded for correctness until after the assignment has been submitted. The remaining problems will be graded for completeness (i.e., does it look like there was a good-faith effort to solve the problem?).

Additional textbook questions for practice (not graded): 3.1, 3.4

Problems:

1. **How far can we extend Dirac's theorem?**

   Let $G = (V, E)$ be a simple undirected graph on $n \geq 3$ vertices such that for every pair of non-adjacent vertices $u, v \in V$ the sum of their degrees is at least n; that is, $d(u) + d(v) \geq n$.

   (a) Show that G has a Hamiltonian cycle. *Hint: Use the proof of Dirac's theorem.*

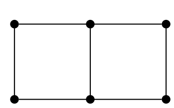
   Notice that the statement above shows a generalization of Dirac's Hamiltonian cycle theorem since $\delta(G) \geq n/2$ implies that the sum of degree of any two vertices is at least n. Let's now also show that this new characterization is tight in the sense that you cannot reduce the degree sum condition:

   (b) For each $n \geq 3$, describe a simple graph $G = (V, E)$ on n vertices satisfying both of the following two conditions:

   - The sum of degrees of all pairs of non-adjacent vertices is at least $n - 1$: that is, for all $u, v \in V$ such that $\{u, v\} \notin E$, we have $d(u) + d(v) \geq n - 1$.
   - G has no Hamiltonian cycle.

2. **Do grid graphs have Hamiltonian cycles?**

   The grid graph $G_{n,m}$ is a simple undirected graph with nm vertices in the set $\{(i, j) : 1 \leq i \leq n, 1 \leq j \leq m\}$. There is an edge between vertices (i, j) and (k, ℓ) if $i = k$ and $|j - \ell| = 1$ or if $j = \ell$ and $|i - k| = 1$. For example, the grid graph $G_{2,3}$ is shown below:



Prove that $G_{n,m}$ does not have a Hamiltonian cycle whenever n and m are both odd. *Hint: Think about other examples of graphs in class that we've shown don't have a Hamiltonian cycle.*

3. **Can you contract a tree into a triangle?**

   Recall the graph contraction operation for an undirected graph G: for a subset of the vertices $X \subseteq V(G)$, we construct the contracted graph G/X by removing all vertices from X in G and adding a new vertex x with edges to all remaining vertices in G that were in the neighborhood of X. Specifically,

   - Vertex set: $V(G/X) = (V(G) - X) \cup \{x\}$.
   - Edge set $E(G/X)$ is the union of the following sets:
     - Edges on the non-contracted vertices: $\{\{u, v\} \in E(G) : u, v \in V(G) - X\}$
     - Edges between x and neighborhood of X: $\{\{x, y\} : y \in N(X)\}$

   Prove/Disprove: A connected graph G is a tree if and only if there is no set $X \subseteq V(G)$ such that G/X has a triangle (i.e., the complete graph on 3 vertices) as a subgraph.

4. **With our (spanning tree) powers combined**

   Suppose that G is a simple undirected graph with two edge-disjoint spanning trees $T_1 = (V, E_1)$ and $T_2 = (V, E_2)$. Edge-disjointness implies that the two trees do not share an edge; i.e., $E_1 \cap E_2 = \emptyset$. Prove that there is a connected spanning subgraph of G where all the vertices have even degree. Recall that a spanning subgraph H of a graph G is a subgraph of G such that $V(H) = V(G)$.

   Strategy: Starting with the spanning tree T_1, consider an iterative process that decreases the number of odd degree vertices in the graph by adding or removing edges from T_2. Hint: At each step of this process, there is a set of remaining vertices that have odd degree. Consider a pair of vertices in this set such that the length of the path between them in T_2 is minimal. What does that imply about the degree of the vertices between them? Can you add/subtract edges from this path so that the vertices now all have even degree?