ON THE CYCLIC VAN DER WAERDEN NUMBERS

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Suppose $k, r \in \mathbb{Z}$ and $k \geq 3$ and $r \geq 2$. The *Van der Waerden number* $W(k; r)$ is the smallest positive integer N such that every coloring of $\{0, \ldots, N-1\}$ with r or fewer colors contains a monochromatic k-term arithmetic progression. That is, there exists a monochromatic subset $\{a, a+d, \ldots, a+(k-1)d\} \subseteq \{0, \ldots, N-1\}$, where $d > 0$. These numbers are well defined by a famous theorem of B. van der Waerden; see [3] for a full account. Very few of these numbers are known exactly. For the purposes of this note, it will suffice to know that $W(3; 2) = 9$, as detailed in [2].

Let the ordinary integers $0, \ldots, N-1$ sometimes stand for their congruence classes mod N. It is hoped that meaning will be clear from context. For instance, in the equation $\mathbb{Z}_N = \{0, \ldots, N-1\}$, $\mathbb{Z}_N$ is, by convention, the set of elements of the ring of integers mod N, and therefore $0, \ldots, N-1$ stand for congruence classes. A *k-term arithmetic progression mod N* is a list $a, a+d, \ldots, a+(k-1)d \in \mathbb{Z}_N$ in which the k terms are *distinct*. For instance, $8, 2, 9$ is a 3-term arithmetic progression mod 13 ($a = 8, d = 7$).

In [1] the *cyclic Van der Waerden number* $W_c(k; r)$ is defined to be the smallest positive integer M such that for all $N \geq M$,

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every coloring of $\mathbb{Z}_N$ with r or fewer colors yields a monochromatic k-term arithmetic progressions mod N in $\mathbb{Z}_N$. Because an ordinary k-term arithmetic progression in $\{0, \ldots, N-1\}$ can be considered to be a k-term arithmetic progression mod N, $W_c(k;r)$ is well-defined by van der Waerden's theorem, and $W_c(k;r) \leq W(k;r)$.

Let us say that a positive integer N is $VDW(k;r)$-*good* if there is a coloring of $\mathbb{Z}_N$ with r or fewer colors such that no k-term arithmetic progression mod N is monochromatic. Thus, $1, \ldots, (k-1)r$ are $VDW(k;r)$-good for all $r \geq 1$, and $W_c(k;r)$ is the next integer after the largest $VDW(k;r)$-good integer. Let $G(k;r) = \{N \mid N \text{ is } VDW(k;r)\text{-good}\}$. An open question left in [1], stated in different terms, is as follows: if $k \geq 3$, $r \geq 2$, is $G(k;r)$ necessarily a block of consecutive integers, $\{1, \ldots, W_c(k;r) - 1\}$? Note that it was the possibility of a "no" answer that necessitated the slight complication in the definition of $W_c(k;r)$ in [1], " ... smallest M such that for all $N \geq M$ ... " rather than " ... smallest N such that ... "

This note will establish that $G(3;2)$ is not a block of consecutive integers; $W_c(3;2)$ will be determined as a corollary. First, we will prove a lemma that will not only help us find $G(3;2)$, but may have applications in further studies of the cyclic Van der Waerden numbers.

Lemma. *Let N be any odd integer greater than or equal to 5. For every 2-coloring of $\mathbb{Z}_N$, $\mathbb{Z}_N$ contains a 3-term monochromatic arithmetic d-progression mod N of the form $a, a+d, a+2d$ where $d \in \{1, \frac{N-1}{2}, \frac{N-3}{2}\}$.*

Proof. For any 2-coloring of $\mathbb{Z}_N$, let $R \subseteq \mathbb{Z}_N$ denote the set of integers of one color and let $B = \mathbb{Z}_N - R$. Because N is odd, we can assume without loss of generality that $|R| > |B|$. Therefore, by the pigeonhole principle, there exists $x \in \mathbb{Z}_N$ such that $\{x, x+1\} \subset R$. If $x-1 \in R$ or $x+2 \in R$ then R contains a 3-term arithmetic 1-progression mod N. If $x + \frac{N+1}{2} \in R$ then $x+1, x+\frac{N+1}{2}, x$ is a 3-term arithmetic $\left[\frac{N-1}{2}\right]$-progression mod N contained in R.

Thus, assume that $\{x-1, x+2, x+\frac{N+1}{2}\} \cap R = \emptyset$. This implies that $\{x-1, x+2, x+\frac{N+1}{2}\} \subseteq B$. However, $x+2, x+\frac{N+1}{2}, x-1$ is a 3-term arithmetic $\left[\frac{N-3}{2}\right]$-progression mod N because $N \geq 5$. $\square$

Theorem. $G(3;2) = \{1, 2, 3, 4, 6, 8\}$

Proof. It is clear that $\{1, 2, 3, 4\} \subseteq G(3;2)$ because $\mathbb{Z}_1, \mathbb{Z}_2, \mathbb{Z}_3$, and $\mathbb{Z}_4$ can all be 2-colored such that no three integers are the same color. Since $W(3;2) = 9$ and $W(3;2) \geq W_c(3;2), \forall N \geq 9, N \notin G(3;2)$. Using the above lemma is it also clear that 5 and 7 are not in $G(3;2)$. It remains to be seen that $6, 8 \in G(3;2)$.

Using G and Y for colors, $GGYYGY$ indicates a 2-coloring of $\mathbb{Z}_6$ which, as may be verified straightforwardly, admits no monochromatic 3-term arithmetic progression mod 6. Similarly, the coloring $GGYYGGYY$ shows that 8 is $VDW(3;2)$-good. $\square$

Corollary. $W_c(3;2) = 9$

Clearly, this shows that the definition of $W_c(k, r)$ as stated in [1] cannot be simplified. However, an open question that remains can be stated as follows: are there any values $k \geq 3, r \geq 2$ such that $G(k;r) = \{1, 2, \ldots, W_c(k;r) - 1\}$?

References

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