Deciding the Winner of an Arbitrary Finite Poset Game is PSPACE-complete

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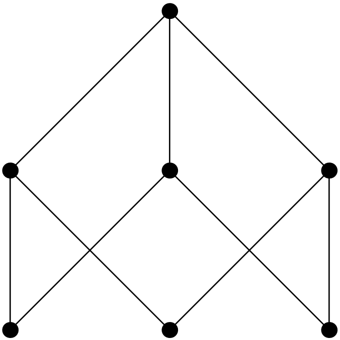
Outline

- Introduce Poset Games
- Positive Results
- Introduce Node Kayles
- Reduce Node Kayles to Poset Games

What is a Poset Game?

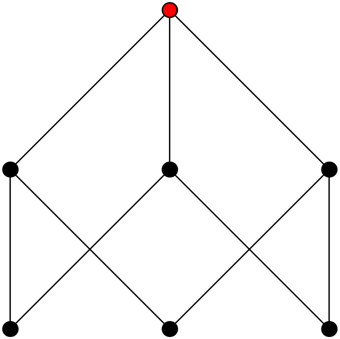
- The game starts with a finite partially ordered set (poset).
- Players take turns choosing an element of the poset, removing it and all elements greater than it.
- The first player unable to move loses.

Poset Game



First Player

Poset Game



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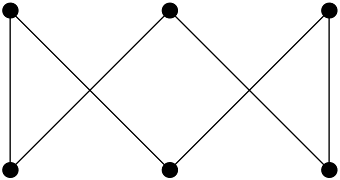
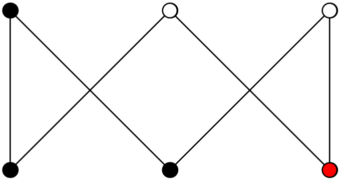


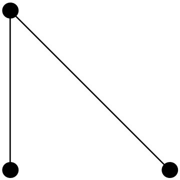
Second Player

Poset Game



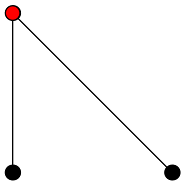
Second Player

Poset Game



First Player

Poset Game



First Player

Poset Game

Second Player

Poset Game

Second Player

Poset Game

First Player

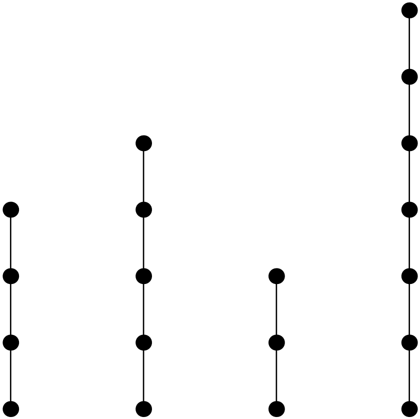
Poset Game

First Player

Poset Game

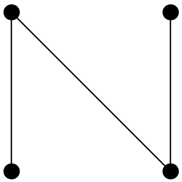
Second Player

Nim [Bouton 1901]



$$100 \quad \oplus \quad 101 \quad \oplus \quad 011 \quad \oplus \quad 111 = 101$$

N-free Poset Games



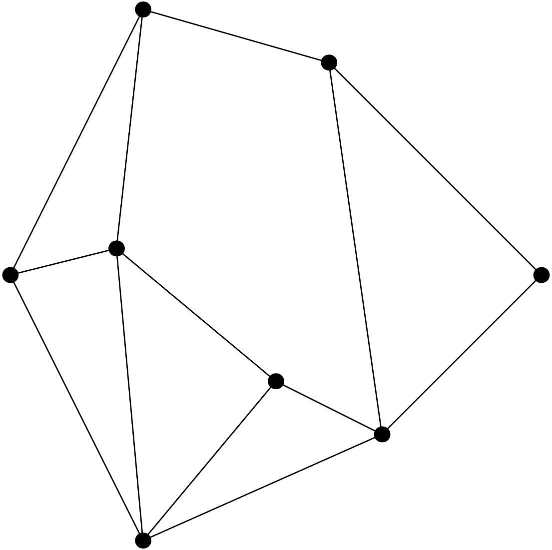
Theorem [Deuber, Thomassé 1996]

There exists a polynomial time algorithm to find the winner of any poset game that does not contain an induced 'N'.

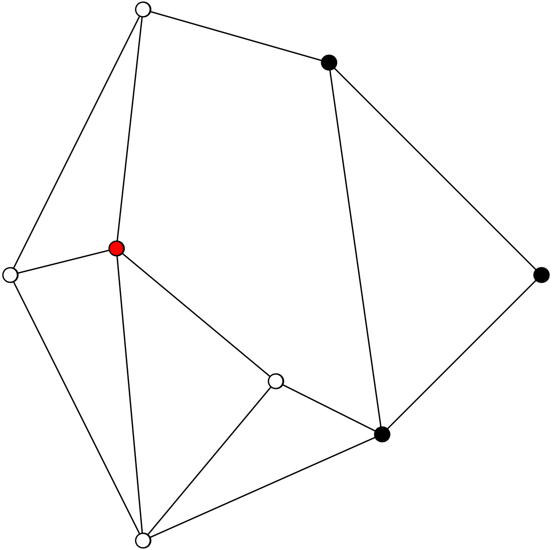
Node Kayles

- The game starts with a simple undirected graph.
- Players take turns choosing a vertex of the graph, removing it and all of its neighbors.
- The first player unable to move loses.

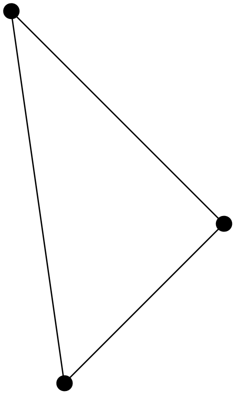
Node Kayles



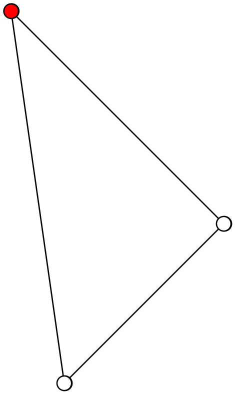
Node Kayles



Node Kayles



Node Kayles



Node Kayles

Node Kayles

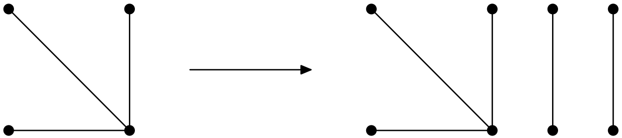
Theorem [Schaefer 1978]

Node Kayles is PSPACE*-complete.*

Reduction Preliminaries

What we want:

1. Number of edges in the Node Kayles graph to be odd.
2. For every vertex in the Node Kayles graph, there is an edge that is not incident to it.



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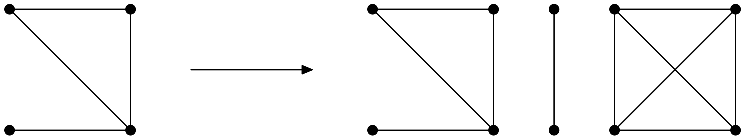
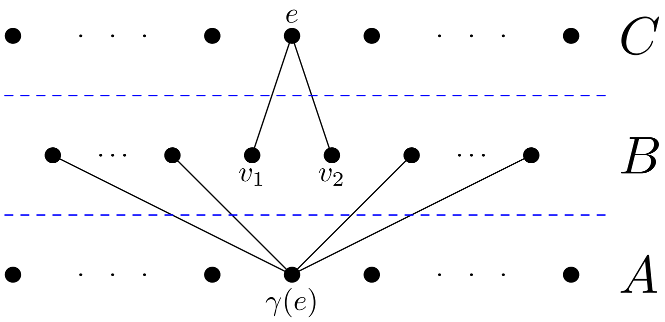


Reduction

Create poset game with three levels $A < B < C$ where

- The elements of B are the vertices of the Node Kayles graph.
- A and C are copies of the edges in the Node Kayles graph.

For each $e = (v_1, v_2)$ edge in the Node Kayles graph add the following edges to poset:



Proof of Correctness

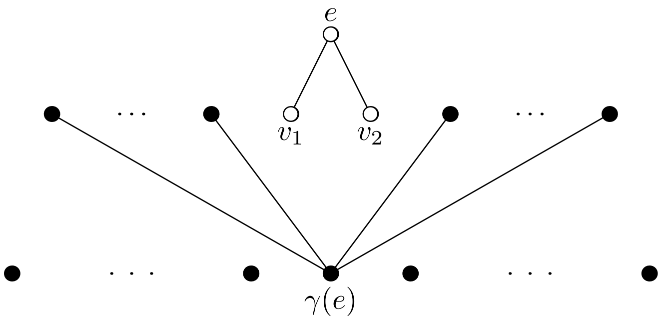
We will argue inductively and assume that no moves in A or C have yet been chosen.

We will call the first player to choose a point in either A or C the *challenger*, making the other player the *responder*.

Proof of Correctness

Lemma 1

If both v_1 and v_2 have been chosen, then $\gamma(e)$ is a winning move.



Parity of bottom: 1

Proof of Correctness

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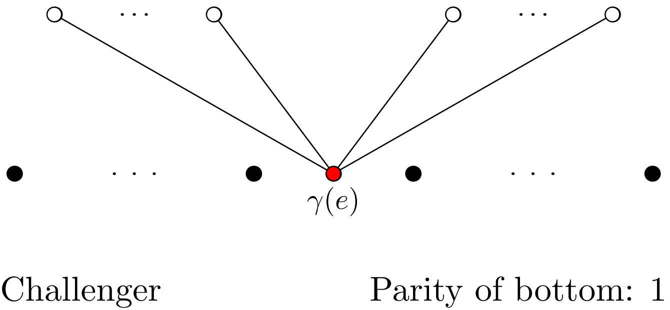

Proof of Correctness

Lemma 1

If both v_1 and v_2 have been chosen, then $\gamma(e)$ is a winning move.

Responder

Parity of bottom: 0

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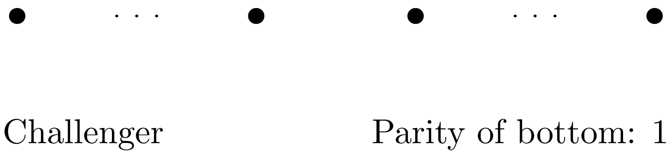

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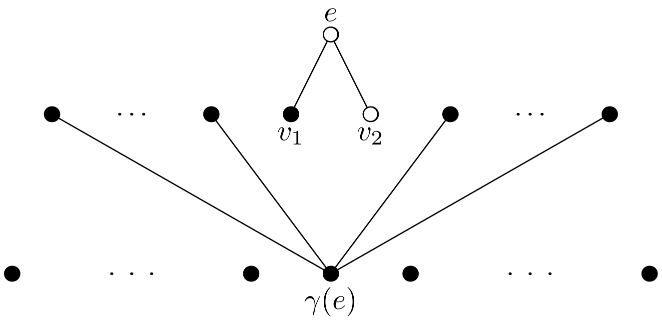
Proof of Correctness

Lemma 2

If exactly one of v_1 and v_2 has been chosen, then $\gamma(e)$ is a losing move.



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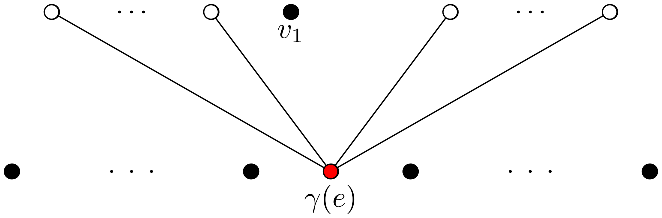

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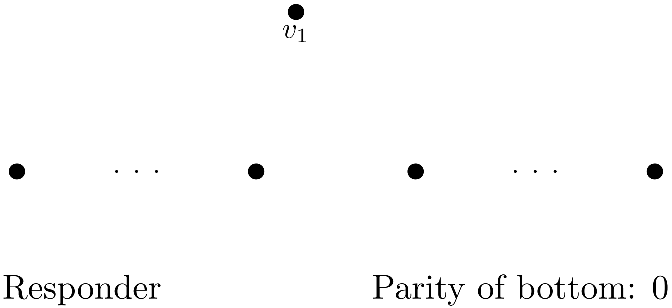

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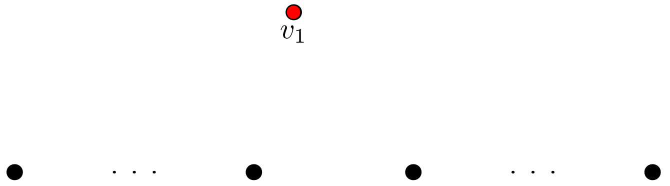

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Challenger

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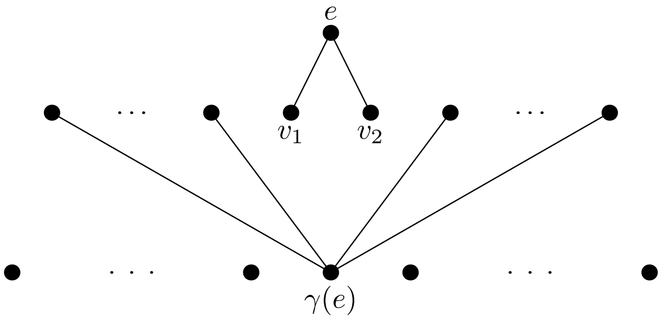
Proof of Correctness

Lemma 3

If neither v_1 nor v_2 has been chosen, then both e and $\gamma(e)$ are losing moves.



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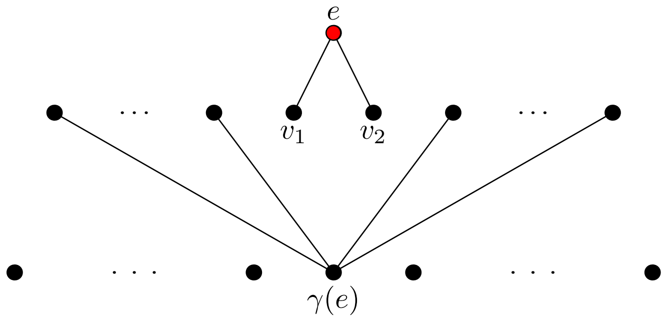

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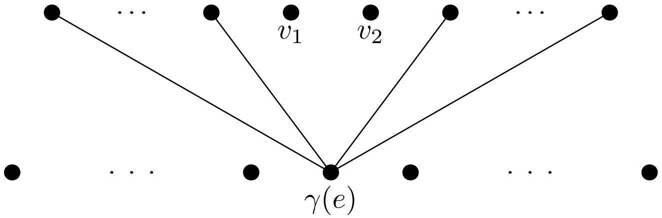

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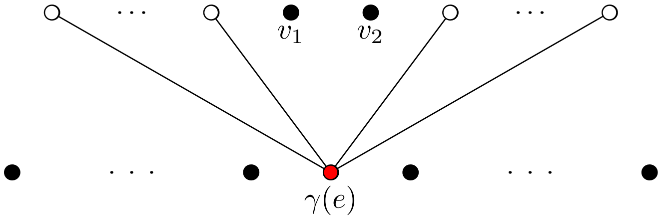

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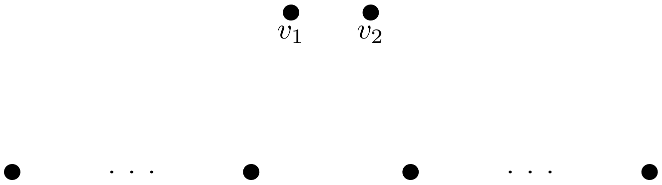

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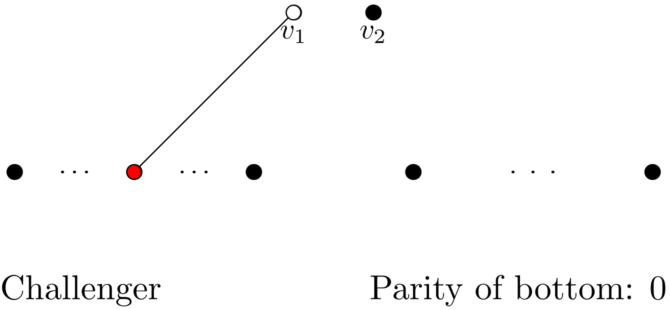

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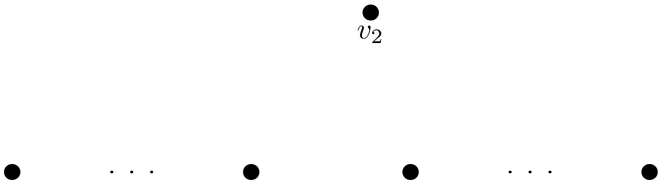

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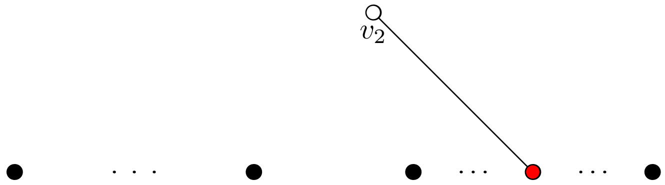

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Recapitulation

- If both v_1 and v_2 have been chosen, then $\gamma(e)$ is a winning move.
- If exactly one of v_1 and v_2 has been chosen, then $\gamma(e)$ is a losing move.
- If neither v_1 nor v_2 has been chosen, then both e and $\gamma(e)$ are losing moves.

Open Questions

- Two Level Poset Games
- Blue-Red Poset Games