Game values and computational complexity: An analysis via black-white combinatorial games

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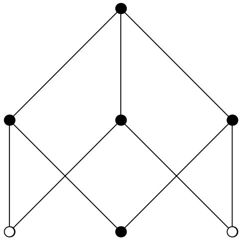
Starting Point - Combinatorial Games

- 2 player perfect information games
- Classics: Chess, checkers, go
- Not-so classic: Nim, Amazons, Geography, Poset Game, Red-Blue Hackenbush, Game of Col

Observation

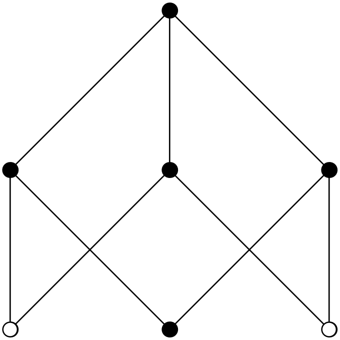
Almost all natural games with a bounded number of moves are either in P *or are* PSPACE*-complete.*

Black-White Poset Games



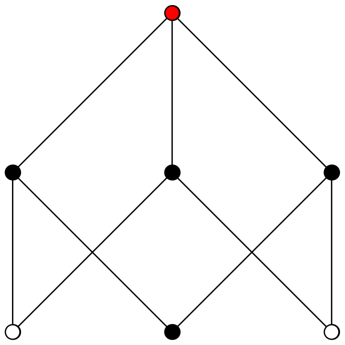
- Players take turns choosing elements from the poset
- Black can only choose black elements
- White can only choose white elements
- The chosen element and all elements greater than it are removed
- First player unable to make a move loses (called *normal gameplay*)

Black-White Poset Game Example



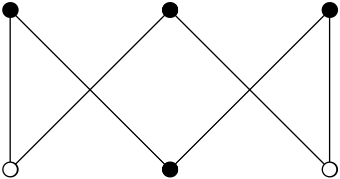
Black

Black-White Poset Game Example



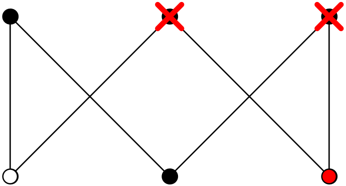
Black

Black-White Poset Game Example



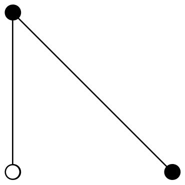
White

Black-White Poset Game Example



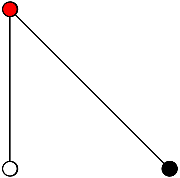
White

Black-White Poset Game Example



Black

Black-White Poset Game Example



Black

Black-White Poset Game Example

White

Black-White Poset Game Example

White

Black-White Poset Game Example

Black

Black-White Poset Game Example

Black

Black-White Poset Game Example

White

Game Values: [Sprague-Grundy 30's], [Conway 76]

- Game value allows you to calculate the winner of games played side by side.
- Intuition: indicates the number of "free" moves for each player.
- If positive, White wins.
- If negative, Black wins.
- If 0, second player wins.
- Other crazy things - Not a total order.

Game values: Black-White Poset Games

4

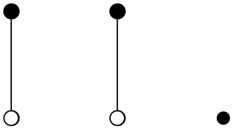
Game values: Black-White Poset Games

$$-3$$

Game values: Black-White Poset Games

$$\frac{1}{2}$$

Game values: Black-White Poset Games

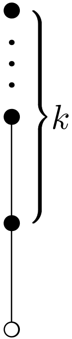


$$\frac{1}{2} + \frac{1}{2} - 1 = 0$$

Game values: Black-White Poset Games

$$\frac{1}{4}$$

Game values: Black-White Poset Games



$$\frac{1}{2^k}$$

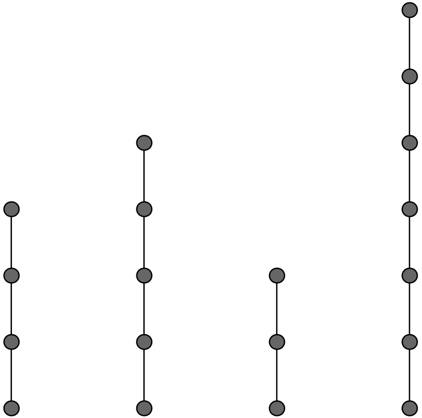
Game values: Poset Games

$$* = \{0 \mid 0\}$$

Game values: Poset Games

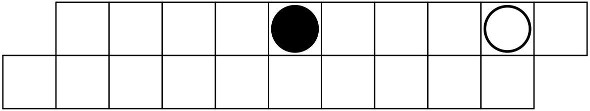
$*4$

Game values: Poset Games



$$*100_2 \;\oplus\; *101_2 \;\oplus\; *011_2 \;\oplus\; *111_2 = *101_2$$

Game values: Amazons



[Snatzke 2004]
{ { 14 | { 11 | { 8 | -8 } , { { 9 | { 4 | { 1 | -1 } } , { 6 | { 3 | -3 } , { { 2 , { 2 | 0 } | 0 , { 1 | { 0 | -1 } ,
-1 } } , 1.5 , { 2 | 0 } | -1 , { 0 | -2 } } } , 3 } , 6 | -6 , { -3 , { { 1 | -1 } | -4 } | -7 } } , { { 9 , { 10 | { 7 |
{ 4 | -4 } , { 5 , { 6 | 4 } | -5 , { * , { 5 | -5 } | -6 } } } , 6 } | -1 } , { 9 , { 10 | { 7 | { 4 | -4 } , { 5 , { 6 |
4 } | -5 , { * , { 5 | -5 } | -6 } } } , 6 } | * , { 3 , { 4 | { 1 | -1 } , { { 2 | 0 } , 1 | -1 , { 0 | -2 } } , { { 0.5
| 0 } , 0 | d } } , { 4 | { 1 | * , 0 } , 0 } | { { * , 0 | -1 } , 0 | -4 } , -3 , { { 1 | -1 } , { 1 , { 2 | 0 } | { 0 |
-2 } , -1 } , { u | { 0 | -0.5 } , 0 } | -4 } } } , { { 10 | { 5 | 3 , { 4 | { 2 | -2 } } } } , 9 | { 4 | { 1 | -1 } } ,
3 } , 6 | -6 , { { 4 | -4 } , { 7 , { 8 | { 5 | { 2 | -2 } , { 1 | -1 } , { 5 | -5 } } , 4 } , { 8 | { 5 | -5 } } | -7 , {
{ { 1 | -1 } , { 5 | -5 } | -5 } , -4 | -8 } , { -2 | -8 } } | -7 } , { -3 , { { 2 | -2 } , { 1 | -1 } | -4 } | -7 } } , {
{ 9 | 3 } , { { 10 | 4 } , 9 | { 6 | { 2 | -2 } } , 3 } , 6 | -6 , { { 4 | -4 } | -7 } , { -3 , { { 1 | -1 } , { 4 | -4 }
| -4 } | -7 } } , { { 9 , { 10 | 8 } | { 8 | { 5 | { 2 | -2 } , { { 1 | 1 } | { * , { 1 , { 2 | 0 } | { 0 | -2 } , -1 } |
-1 } } , { { { 1 | 1 } , 1 | 1 } | { * , { 1 , { 2 | * , { 1 | -1 } } | -1 , { 0 | -2 } } | -1 } } } , 4 } , 5 , { 6 | 4 ,
{ 5 | { { 1 | 1 } , { 2 | 0 } | { * , { { 2 | 0 } , 1 | -1 , { 0 | -2 } } | -1 } } , { { { 1 | 1 } , 1 | 1 } , { 2 | 0 ,
{ 1 | -1 , { { 0.5 | 0 } , 0 | -1 , { 0 | -2 } } } } } | { * , { 1 , { 2 | * , { 1 | -1 } } | -1 , { 0 | -2 } } | -1 } }

Game values: Amazons

, { 2 | -2 } } } } , { 9 | 7 } , { { 10 | 8 } | 7 } | -6 , { { 3 | -3 } | -7 } } } , 10 , { 11 | { 8 | -8 } , { { { 9 | { 5 | { 2 | -2 } , { { 4 | 0 } , { 2 | 1 } | { * | -1 } }{ , { { 4 | 0 , { { 2 | 0 } , 1 | -1 , { 0 | -2 } } } , { { 5 | 1 } , 2 | 1 } | { * | -1 } } } , { 6 | { 3 | -3 } , { { 2 , { 2 | 0 } | 0 , { 1 | { 0 | -1 } , -1 } } , 1.5 , { 2 | 0 } | -1 , { 0 | -2 } } } , 4 } , 7 , { 8 | 4 , { 5 | { { 4 | 0 } , { 2 | 1 } | { * | -1 } } , { { 4 | 0 , { { 2 | 0 } , 1 | -1 , { 0 | -2 } } } , { { 5 | 1 } , 2 | 1 } | { * | -1 } } , { 2 | -2 } } } | { 0.5 , { 1 | * } | { -1 | -3 } , { { { 1 | 1 } | -2 } , 0.25 | -5 , { { { { 2 | { 1 | -1 } , * } , 1 | -1 , { 0 | -2 } } , * | -3 } , -2 | -6 } } } , { 0 | -0.5 } } , { 9 | { 4 | { 1 | -1 } } , { 6 | { 3 | -3 } , { { 2 , { 2 | 0 } | 0 , { 1 | { 0 | -1 } , -1 } } , 1.5 , { 2 | 0 } | -1 , { 0 | -2 } } } , { 5 | { 2 | -2 } , { { 4 | 0 } , { 2 | 1 } | { * | -1 } } , { { 4 | 0 , { { 2 | 0 } , 1 | -1 , { 0 | -2 } } } , { { 5 | 1 } , 2 | 1 } | { * | -1 } } } , 3 } , 6 | -6 , { -3 , { { 1 | -1 } | -4 } | -7 } } , { { { 9 | { 5 | { 2 | -2 } , { { 4 , { 5 | { 2 | * , { 1 | -1 } , { 2 | -2 } } , 1 } | 0 , { 1 , { 2 | 0 } | { 0 | -2 } , -1 } } , { { 5 | 1 } , 2 | 1 } | * , { * | -1 } } } , { 6 | { 1.5 , { 2 , { 2 | 0 } | { 1 | { 0 | -1 } , -1 } , 0 } , { 2 | 0 } | -1 , { 0 | -2 } } , { 3 | -3 } } , 4 } , 7 , { 8 | 4 , { 5 | { 1 , { 3 | { 0 | -0.5 } } , { { 2 | 1 } , 2 | 0 } | -1 } , { { 4 , { 5 | 1 } | 0 , { { 2 | 0 } , 1 | -1 , { 0 | -2 } } } , { { 5 | 1 } , 2 | 1 } | { * | -1 } } , { 2 | -2 } } } | { 0.5 , { 1 | * } | { -1 | -3 } , { { { 1 | 1 } | -2 } , 0.25 | -5 , { { { { 2 | { 1 | -1 } , * } , 1 | -1 , { 0 | -2 } } , * | -3 } , -2 | -6 } } } , { 0 | -0.5 } } , { 9 | { 4 | { 1 | -1 } } , { 6 | { 1.5 , { 2 , { 2 | 0 } | { 1 | { 0 | -1 } , -1 } , 0 } , { 2 | 0 } | -1 , { 0 | -2 } } , { 3 | -3 } } , { 5 | { 2 | -2 } , { { 4 , { 5 | { 2 | * , { 1 | -1 } , { 2 | -2 } } , 1 } | 0 , { 1 , { 2 | 0 } | { 0 | -2 } , -1 } } , { { 5 | 1 } , 2 | 1 } | * , { * | -1 } } } , 3 } , 6 | -6 , { -3 , { * , { 1 | -1 } | -4 } | -7 } } , { { 9 , { 10 | { 7 | { 4 | -4 } , { 5 , { 6 | 4 } | -5 , { * , { 5 | -5 } | -6 } } } , 6 } | -1 } , { { 9 | { 6 | { 3 | -3 } } , 5 } , 8 | -2 } , { 9 , { 10 | { 7 | { 4 | -4 } , { 5 , { 6 | 4 } | -5 , { * , { 5 | -5 } | -6 } } } , 6 } | * , { 3 , { 4 | { 1 | -1 } , { { 2 | 0 } , 1 | -1 , { 0 | -2 } } , { { 0.5 | 0 } , 0 | d } } , { 4 | { 1 | * , 0 } , 0 } | { { * , 0 | -1 } , 0 | -4 } , -3 , { { 1 | -1 } , { 1 , { 2 | 0 } | { 0 | -2 } , -1 } , { u | { 0 | -0.5 } , 0 } | -4 } } } , { { 9 | { 6 | { 3 | -3 } } , 5 } , 8 | { 4 , { 5 | { 2 | { { 1 | * } , 0.5 | { * | -1 } , -0.5 } , 0 } , 1 } | { { { 3 | { { 1 | * } , 0.5 | { * | -1 } , -0.5 } , * } , 2 | -2 , { { { 1 | * } , 0.5 | { * | -1 } , -0.5 } , * | -3 } } , { 2 | -2 }

Game values: Amazons

* 3 more slides *

Game values: Amazons

, * } , 2 | -2 , { { { 1 | * } , 0.5 | { * | -1 } , -0.5 } , * | -3 } } , { 2 | -2 } , { { 1 | { 1 | -1 } } | { * , 0 | { 0
| -2 } , -1 } , -1 } | -5 } , -4 , { { 0 , { { 1 | * } , 0.5 | { * | -1 } , -0.5 } | -2 } , -1 | -5 } } , { 1 | -1 } } , 3
} | 3 } , { { { 6 | 6 } | 3 } , { { 6 | 6 } , { 6.75 | 4 } | 1 , { 2 | 0 } } | * } , { 6 , { 7 | { 4 | { { 5 | { 2 , {
3 | { { 1 | * } , 0.5 | { * | -1 } , -0.5 } , * } | { { { 1 | * } , 0.5 | { * | -1 } , -0.5 } , * | -3 } , -2 } , { 2 | -2
} , { { { 2 | 0 } , 1 | * , 0 } , 1 | { { 1 | -1 } | -1 } } } , 4 , { 5 | { 2 | 0 , { { 1 | * } , 0.5 | { * | -1 } , -0.5
} } , 1 } | -4 , { { { { 1 | * } , 0.5 | { * | -1 } , -0.5 } , 0 | -2 } , -1 | -5 } } , { 1 | -1 } } , 3 } | { 1 | -1 , {
0 | -4 } } , -0.5 } | { -1 , { 0 | -2 } | -2 } } , { 2 | { 1 | { { 0 , { 4 | 0 } | 0 } | -1 } , { 0 | { { * , { 1 , { 2 |
* , { 1 | -1 } } | -1 , { 0 | -2 } } | -3 } , -2 | -6 } , -5 , { { { { 3 | { { 1 | * } , 0.5 | { * | -1 } , -0.5 } , * } ,
2 | -2 , { { { 1 | * } , 0.5 | { * | -1 } , -0.5 } , * | -3 } } , { 1 | -1 } | -4 } , -3 | -7 } } , { { { 2 | 0 } , { 0 |
{ 0 | -0.5 } } | -3 } , { 0.5 , { 1 , { 1 | 1 } | 0 } , { 2 | -0.5 } | -0.5 } , 0 , { { 1 , { 2 | 0 } | -0.25 } , { 2 | 0
} , 0 | -1 } | -6 , { { { 1 | -1 } , { 2 , { 3 | { { 1 | * } , 0.5 | { * | -1 } , -0.5 } , * } | { { { 1 | * } , 0.5 | { *
| -1 } , -0.5 } , * | -3 } , -2 } , 0 | -4 } , -3 | -7 } } , -1 } , -0.75 } , { 2 | 0 , { 1 | { 0 | { { * , { { 2 | 0 } , 1
| -1 , { 0 | -2 } } | -3 } , -2 | -6 } , -5 , { { { { 3 | { { 1 | * } , 0.5 | { * | -1 } , -0.5 } , * } , 2 | -2 , { { { 1
| * } , 0.5 | { * | -1 } , -0.5 } , * | -3 } } , { 1 | -1 } | -4 } , -3 | -7 } } , { { { 2 | 0 } , { 0 | { 0 | -0.5 } } |
-3 } , { 0.5 , { 1 , { 1 | 1 } | 0 } , { 2 | -0.5 } | -0.5 } , 0 , { { 1 , { 2 | 0 } | -0.25 } , { 2 | 0 } , 0 | -1 } | -6
, { { { 1 | -1 } , { 2 , { 3 | { { 1 | * } , 0.5 | { * | -1 } , -0.5 } , * } | { { { 1 | * } , 0.5 | { * | -1 } , -0.5 } ,
* | -3 } , -2 } , 0 | -4 } , -3 | -7 } } , -1 } } , 1 }

You might think...

Consider some classic problems in combinatorial game theory with unknown complexity:

- Red-Blue Hackenbush - game value is only a "number"
- Game of Col - game value is only a "number" $+ *$

Conjecture

"Simplicity" of game value is barrier to PSPACE*-completeness.*

Simple Construction

- Take any PSPACE-complete game
- Modify the rules so that the players *must* alternate
- Game can only take the values -1, 0, or 1

What now?

- What about natural games?
- Red-Blue Hackenbush, Col?

Results - PSPACE-**completeness:**

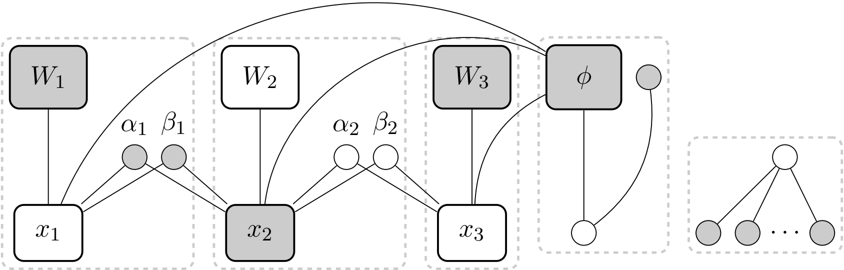
- Black-White Poset Games - game value is a “number”
- Game of Col - game value is either 0 or $*$

Black-White Poset Games are PSPACE-complete

TQBF:

$$\exists x_1 \forall x_2 \exists x_3 \cdots \exists x_{2n-1} \forall x_{2n} \exists x_{2n+1} \phi(x_1, x_2, \ldots, x_{2n+1})$$

where $\phi = c_1 \wedge c_2 \wedge \cdots \wedge c_m$



Open Questions

- What is the complexity of Red-Blue Hackenbush?
- What are the meaningful properties of a game that dictate its complexity?