

Instructions: There are two parts to this homework:

- Concept check (Problem 1): Complete individually on Gradescope.
- Written Homework (Problems 2–3): You may work individually or in a team of up to 3 people. If working in a group, make only one submission per group on Gradescope: one person uploads the submission and then adds the other group members to the submission by selecting their names in the “Add Group Members” dialog box. All problems will be graded for completeness. Completeness means that it looks like there was a good-faith effort to solve the problem. A LaTeX template for the homework is provided on Canvas.

Problems:

1. Concept check

Complete the assignment “Homework 2 - Concept Check” on Gradescope.

2. Dirac notation and tensor products

Recall that we use $|0\rangle$ to denote the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and $|1\rangle$ to denote the vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Therefore, we can express every qubit as

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$$

where $\alpha, \beta \in \mathbb{C}$ such that $|\alpha|^2 + |\beta|^2 = 1$. In other words, $|\psi\rangle$ is the vector $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$. We have the following basic single-qubit unitary matrices:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

- Let $|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$ be any single-qubit quantum state. Write the following states as a linear combination of $|0\rangle$ and $|1\rangle$: $H|\psi\rangle$, $X|\psi\rangle$, and $HZH|\psi\rangle$. Based on these computations, what identity between single-qubit operators can you prove?
- Recall that a two-qubit state is *entangled* if it cannot be written as the tensor product of two single-qubit states. Give an example of a 2-qubit operation U such that $U(|0\rangle \otimes |0\rangle)$ is entangled. (Make sure to check that U is unitary!)

Recall that for n -qubit unitary U and m -qubit unitary V we define their tensor product $U \otimes V$ to be the unique $2^{m+n} \times 2^{m+n}$ matrix satisfying the equation

$$(U \otimes V)(|\psi\rangle \otimes |\varphi\rangle) = (U|\psi\rangle) \otimes (V|\varphi\rangle) \tag{1}$$

for all n -qubit states $|\psi\rangle$ and m -qubit states $|\varphi\rangle$.

- (c) Let $U = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$ and $V = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}$ be arbitrary 2×2 unitary matrices. Write out the 4×4 tensor product matrix $U \otimes V$.
- (d) Using the characterization in Equation 1, prove that what you wrote down in the previous part is correct. *Hint: A matrix is uniquely determined by its action on a set of basis states. A basis for two-qubit states is $\{|0\rangle \otimes |0\rangle, |0\rangle \otimes |1\rangle, |1\rangle \otimes |0\rangle, |1\rangle \otimes |1\rangle\}$.*

3. Inner products and bases

Let $|\psi\rangle$ be an arbitrary quantum state. In other words, we can write

$$|\psi\rangle = \sum_{x \in \{0,1\}^n} \alpha_x |x\rangle$$

with $\sum_{x \in \{0,1\}^n} |\alpha_x|^2 = 1$. Recall, $|\psi\rangle$ is a column vector. We define the conjugate transpose of $|\psi\rangle$ as the row vector

$$\langle\psi| = \sum_{x \in \{0,1\}^n} \alpha_x^* \langle x|.$$

We define the *inner product* of two quantum states $|\psi\rangle = \sum_{x \in \{0,1\}^n} \alpha_x |x\rangle$ and $|\varphi\rangle = \sum_{x \in \{0,1\}^n} \beta_x |x\rangle$ as

$$\langle\psi|\varphi\rangle := \langle\psi| |\varphi\rangle = \sum_{x \in \{0,1\}^n} \alpha_x^* \beta_x$$

- (a) Let $|\psi_1\rangle = \frac{|0\rangle + i|1\rangle}{\sqrt{2}}$, $|\psi_2\rangle = \frac{|0\rangle - i|1\rangle}{\sqrt{2}}$, and $|\psi_3\rangle = \frac{|0\rangle + \sqrt{2}|1\rangle}{\sqrt{3}}$. Compute the following inner products: $\langle\psi_1|\psi_2\rangle$, $\langle\psi_1|\psi_3\rangle$, $\langle\psi_2|\psi_3\rangle$.
- (b) Prove that for any n -qubit state $|\psi\rangle = \sum_{x \in \{0,1\}^n} \alpha_x |x\rangle$, we have $\langle\psi|\psi\rangle = 1$.
- (c) Let $|\psi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle$ and $|\varphi\rangle = \beta_0 |0\rangle + \beta_1 |1\rangle$ be arbitrary quantum states such that $\langle\psi|\varphi\rangle = 0$. Give the unitary U such that $U|\psi\rangle = |0\rangle$ and $U|\varphi\rangle = |1\rangle$. Conclude that we can use the unitary U to determine whether or not some unknown state is $|\psi\rangle$ or $|\varphi\rangle$.

An orthonormal basis for n -qubit quantum state vectors is a set of 2^n quantum states $\{|\psi_i\rangle\}_{i=1}^{2^n}$ with the following property: $\langle\psi_i|\psi_j\rangle = 0$ for all $i \neq j$. As a standard linear-algebraic consequence, we can write *every* n -qubit state $|\varphi\rangle$ as a linear combination of those states:

$$|\varphi\rangle = \sum_{i=1}^{2^n} \alpha_i |\psi_i\rangle$$

for some complex amplitudes $\alpha_i \in \mathbb{C}$. The *computational basis* is the orthonormal basis consisting of the basis vectors $\{|x\rangle\}_{x \in \{0,1\}^n}$. For the following problems, let $\{|\psi_i\rangle\}_{i=1}^{2^n}$ be an arbitrary orthonormal basis for n -qubit quantum states.

- (d) Show that the basis expansion, $|\varphi\rangle = \sum_{i=1}^{2^n} \alpha_i |\psi_i\rangle$, is unique. Specifically, imagine some other expansion $|\varphi\rangle = \sum_{i=1}^{2^n} \beta_i |\psi_i\rangle$, and show that $\alpha_i = \beta_i$ for all i . *Hint: use inner products.*

- (e) Given that $|\varphi\rangle = \sum_{i=1}^{2^n} \alpha_i |\psi_i\rangle$, show that $\sum_{i=1}^{2^n} |\alpha_i|^2 = 1$. In other words, the sum of the amplitude magnitudes squared equals 1 regardless of which basis in which we write $|\varphi\rangle$. We have only defined this property in the computational basis.
- (f) Show that for any n -qubit unitary U , $\{U |\psi_i\rangle\}_{i=1}^{2^n}$ is also an orthonormal basis.
- (g) Let $|+\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$ and $|-\rangle = \frac{|0\rangle-i|1\rangle}{\sqrt{2}}$. Write the state $\frac{|0\rangle+i|1\rangle}{\sqrt{2}}$ in the $\{|+\rangle, |-\rangle\}$ basis.