

Instructions: There are two parts to this homework:

- Concept check (Problem 1): Complete individually on Gradescope.
- Written Homework (Problems 2–3): You may work individually or in a team of up to 3 people. If working in a group, make only one submission per group on Gradescope: one person uploads the submission and then adds the other group members to the submission by selecting their names in the “Add Group Members” dialog box. All problems will be graded for completeness. Completeness means that it looks like there was a good-faith effort to solve the problem. A LaTeX template for the homework is provided on Canvas.

Problems:

1. Concept check

Complete the assignment “Homework 3 - Concept Check” on Gradescope.

2. No-Cloning Theorem with junk

In class, we proved the following theorem: there is no two-qubit unitary U for which

$$U(|\psi\rangle \otimes |0\rangle) = |\psi\rangle \otimes |\psi\rangle$$

for all single-qubit states $|\psi\rangle$. However, this is not the only possible definition of a cloning unitary. For example, we might have a 3-qubit cloning unitary U such that

$$U(|\psi\rangle \otimes |0\rangle \otimes |0\rangle) = |\psi\rangle \otimes |\psi\rangle \otimes |\text{junk}(\psi)\rangle$$

for all single-qubit states $|\psi\rangle$ and some arbitrary state $|\text{junk}(\psi)\rangle$ that depends on the cloned state $|\psi\rangle$. In other words, in this setting, the unitary copies the first qubit into the second qubit, but also produces some arbitrary state in the third qubit. One can check that the exact proof we saw in class does not directly apply, so let’s try something else.

Suppose there exists such a cloner U . Recall that $|+\rangle := \frac{|0\rangle + |1\rangle}{\sqrt{2}}$. We can use the definition of the cloner U to show that

$$U(|+\rangle \otimes |0\rangle \otimes |0\rangle) = |+\rangle \otimes |+\rangle \otimes |\text{junk}(+)\rangle. \quad (1)$$

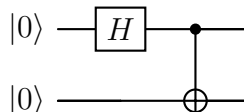
Alternatively, we can write $U(|+\rangle \otimes |0\rangle \otimes |0\rangle)$ using the linearity of matrix-vector multiplication as

$$U\left(\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) \otimes |00\rangle\right) = U\left(\frac{|000\rangle + |100\rangle}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}(U|000\rangle + U|100\rangle) \quad (2)$$

Using the definition of the cloner U on states $|000\rangle$ and $|100\rangle$, show that the state derived in Equation (1) is different from the state in Equation (2). Conclude that cloning states with junk is impossible. (Note: You need to prove that the two states are different.)

3. Circuit building

In this problem, you will practice building a quantum circuit for a quantum state. For example, the circuit which constructs the state $\frac{|00\rangle+|11\rangle}{\sqrt{2}}$ is the following:



Some of you may find it useful to play around with quantum circuits in the Qiskit SDK. Information on how to install Qiskit is [here](#) with tutorials [here](#). *Important:* The ordering of qubits in Qiskit is the opposite of the one that we use in this class (see [here](#)). The Python code below prints the state vector associated with the circuit above:

```
from qiskit import QuantumCircuit
from qiskit.quantum_info import Statevector

qc = QuantumCircuit(2)
qc.h(0)
qc.cx(0,1)

print(qc.draw())
print(Statevector(qc))
```

For each of the following quantum states, draw a quantum circuit which constructs it. Assume that all qubits are initialized to $|0\rangle$. The circuit above was drawn in LaTeX using the [quantikz package](#), but it is also fine to draw the circuit by hand. You may use the following single-qubit gates:

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \quad RY = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & -\sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix}$$

In Qiskit, the final gate is `RYGate(1.91063)` where the angle of rotation comes from the fact that $2 \arcsin(\sqrt{2}/3) \approx 1.91063$. You are also allowed to use the singly or doubly controlled versions of the gates above. Recall that controlled- X is the CNOT gate and controlled-controlled- X is the Toffoli gate.

- (a) $\frac{|000\rangle+|111\rangle}{\sqrt{2}}$
- (b) $\frac{|00\rangle+i|01\rangle+|10\rangle+i|11\rangle}{2}$
- (c) $\frac{|011\rangle+|101\rangle+|110\rangle}{\sqrt{3}}$.

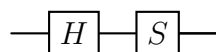
4. Challenge question: Clifford circuits are not universal

This question is optional and will not be graded

The quantum operations that can be built using circuits of Hadamard (H), Phase (S), and CNOT gates are called the *Clifford operations*. While the Clifford operations are

extremely important in quantum computation (we will see many uses in this class), it turns out that they also are actually quite limited. We say that a set of gates is *exactly universal* for a set of unitary operations if for every unitary in the set, there is a quantum circuit which constructs it up to a complex scalar factor of magnitude 1.

For example, Clifford circuits are exactly universal for the set containing the single unitary $\frac{1}{\sqrt{2}} \begin{pmatrix} -1 & -1 \\ -i & i \end{pmatrix}$ since the circuit



constructs the unitary up to a -1 scalar factor—that is, $SH = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$.

- (a) Show that Clifford circuits are exactly universal for the Pauli operators:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Hint: Multiply two of the matrices above. What do you get? Also, use the identity that you proved in the previous homework: $HZH = X$.

- (b) Show that the Clifford circuits are *not* exactly universal for the set of all two-qubit unitary operations. *Hint: Show that each entry of the unitary matrix of a Clifford circuit has a special form. Since the entries of an arbitrary unitary matrix do not have that form, reach the conclusion.*