CSE 200 - Computability and Complexity
Homework 4
Due Wednesday, November 29th, 11:59pm

Instructions: You may work individually or in a team of 2 people. You may switch teams for different assignments. Please ensure your name(s) and PID(s) are clearly visible on the first page of your submission, and then upload the PDF to Gradescope. If working in a group, submit only one submission per group: one partner uploads the submission through their Gradescope account and then adds the other group member to the Gradescope submission by selecting their name in the "Add Group Members" dialog box. You will need to re-add your group member every time you resubmit a new version of your assignment.

It is highly recommended (though not required) that you type your answers. It is your responsibility to make any handwriting clear and legible for grading. For ease of grading, please start each new problem on a separate page.

Problems:

1. **The power of gate sets in constant-depth circuits**

   For families of bounded fan-in circuits, it suffices to consider some canonical gate set like AND, OR, and NOT. Changing the gate set does not change the computational power of the circuit. However, when we allow unbounded fan-in gates—i.e., gates that can have any number of inputs—the choice of our gate set *is* important.

   For every n, define the following gates acting on n Boolean inputs:

$$\text{PARITY}(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} x_i \pmod 2$$
$$\text{MAJ}(x_1, x_2, \ldots, x_n) = \left\lfloor \frac{1}{2} + \frac{1}{n}\sum_{i=1}^{n} x_i \right\rfloor$$

   In other words, the PARITY function outputs 1 if there are an odd number of inputs which are 1, and the MAJ function (short for "majority") outputs 1 if there are there are at least as many 1 inputs as 0 inputs.

   In class, we encountered the complexity class AC^0, which contains those languages computable by constant-depth, poly-size circuits with unbounded fan-in AND, OR, and NOT gates. Let's define the analogous class LC^0 for circuits with PARITY gates and NOT gates, and TC^0 for circuits with MAJ gates and NOT gates. In all classes above, the circuits have constant depth, are polynomial size, and have gates that can act on any number of inputs (except NOT, which always acts on a single input).

   (a) Show that AC^0 and LC^0 are incomparable. That is, $\mathsf{AC}^0 \not\subseteq \mathsf{LC}^0$ and $\mathsf{LC}^0 \not\subseteq \mathsf{AC}^0$. *Hint:* The output of the PARITY gate is a polynomial (see the explicit formula above). The output of the NOT gate can also be expressed as a (certain kind of) polynomial. What does this mean about the output of an LC^0 circuit?

   (b) Show that TC^0 contains both AC^0 and LC^0. *Hint:* A useful step is to have wires in the circuit which are always 1. How can you construct such a wire?

2. **Random Approximate 3SAT**

We know from the Cook-Levin theorem that 3SAT is NP-complete. Therefore, it is unlikely that there is a polynomial-time algorithm to output a satisfying solution to a 3SAT formula (if it exists). Indeed, it is believed that $\mathsf{BPP} \neq \mathsf{NP}$, so it's also unlikely that there is a randomized algorithm could output a satisfying assignment with high probability.

For this problem, let's relax the notion that an assignment to a 3SAT formula needs to satisfy all of the clauses. Instead, let's assume that it only needs to satisfy 6/7ths of all clauses.

Your goal for this problem is to design a polynomial-time randomized algorithm for the following problem: given as input a 3SAT formula $\varphi = C_1 \wedge C_2 \wedge \cdots \wedge C_m$ where each clause has exactly three literals (corresponding to three distinct variables), output a variable assignment satisfying at least $\lfloor 6m/7 \rfloor$ clauses with probability at least 99%.

Hint: Don't try to be too clever with your variable assignment. Let the randomness in your algorithm do the work.

You may find the following concepts useful for analyzing the correctness of your algorithm:

- **Discrete random variable:** A random variable X has some value $n \in \mathbb{Z}$ with probability $\Pr[X = n]$. The sum of all probabilities is 1: $\sum_n \Pr[X = n] = 1$.
- **Expectation:** The expectation of X is $\mathbb{E}[X] = \sum_n n \cdot \Pr[X = n]$. Expectation has an important property called linearity: for any random variables X and Y, $\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$.
- **Markov's inequality:** For any $t \geq 0$ and any random variable X which is always positive (the probability of a negative value is zero), we have $\Pr[X \geq t] \leq \mathbb{E}[X]/t$.
- **Independence:** Two random variables X,Y are independent if the value of one variable does not affect the value of the other variable. That is, for all values $x, y \in \mathbb{Z}$, we have $\Pr[X = x \text{ and } Y = y] = \Pr[X = x] \cdot \Pr[Y = y]$.