CSE 291 / Math 277A - Quantum Complexity Theory (Fall 2025)
Homework 5
Due Friday, November 21, 11:59pm

Instructions: Note: It is highly recommended (though not required) that you type your answers. It is your responsibility to make any handwriting clear and legible for grading. You may work with 1-2 other collaborators, but you must write the solutions separately and clearly mark the names of all people you worked with on each problem.

Problems:

1. **A majority of quantum query techniques**

   For any bitstring $x \in \{0,1\}^N$, we define its majority as the bit that occurs most often:

   $$\mathsf{Maj}(x) = \begin{cases} 1 & \text{if } |x| \geq N/2 \\ 0 & \text{otherwise} \end{cases}$$

   where $|x|$ is the Hamming weight of x.

   Let's first establish that there are no efficient query algorithms for Majority by a reduction from the Or function:

   (a) Recall that $\mathsf{Or}(x) = x_1 \vee \cdots \vee x_N$ has quantum query complexity $\Omega(\sqrt{N})$ by the Grover lower bound. Show that a quantum query algorithm for Majority implies a query algorithm for Or, and therefore, conclude the quantum query complexity of Majority is $\Omega(\sqrt{N})$.

   In fact, Majority is even harder than Or.

   (b) Show that the quantum query complexity of Majority is $\Omega(N)$ by the polynomial method. You will probably want to use the following theorem:

   Theorem ([Paturi 92])**.** *Let $p\colon \mathbb{R} \to \mathbb{R}$ be a real polynomial. Suppose $p(z) \in [0,1]$ on all integer points $z \in \{0,1,\ldots,N\}$. Then, there exists a universal constant $C > 0$ such that*

   $$\deg(p) \geq \max_{z \in [0,N]} \left(\frac{|p'(z)|}{C(1+|p'(z)|)} \sqrt{z(N-z)} \right)$$

2. **Project precursor problems**

   The purpose of this problem is to practice generating research ideas in quantum complexity theory. Concretely, your goal is to write down (at least) 2 research questions that you don't know the answer to. You should write these questions with the intention that one of them may become the basis for your final project in this class (you will repeat this exercise on future homework).

   For each question, you should pick some topic/theorem/area that we've learned about in class and propose some way to extend it. To reiterate—the purpose is to practice coming up with interesting *questions*, not necessarily answers. You also shouldn't yet worry about whether or not your question has been already answered somewhere in the quantum literature.