Math 154 - Discrete Math and Graph Theory
Homework 1
Due **Monday, January 13th, 11:59pm**

Instructions: There may be opportunities to work in groups for future assignments, but since these initial assignments are the basis for all future work in this class, it is important that it is completed individually.

It is highly recommended (though not required) that you type your answers. It is your responsibility to make any handwriting clear and legible for grading. A LaTeX template for the homework is provided on Canvas.

We will only be grading some of the problems below for correctness. However, because all of the concepts are important, we will not reveal which problems are being graded for correctness until after the assignment has been submitted. The remaining problems will be graded for completeness (i.e., does it look like there was a good-faith effort to solve the problem?).

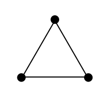
Additional textbook questions for practice (not graded): 1.1, 1.9

Problems:

1. **Is this a real graph?**

   For each list of numbers below, determine if there is a simple graph with that list as its degree sequence. If the answer is "yes", give an explicit graph with that degree sequence. If the answer is "no", prove that no such simple graph has that degree sequence.

   For example, the answer for the sequence $(2, 2, 2)$ is "yes" because the triangle graph has three vertices, each of which has degree 2:

   (a) $(1, 1, 1, 1, 1, 1)$

   (b) $(2, 2, 2, 1, 1, 1)$

   (c) $(5, 3, 3, 3, 2, 2)$

   (d) $(5, 4, 3, 2, 1, 1)$

2. **Every source must lead to a sink**

   Let $\vec{G} = (V, \vec{E})$ be a (finite) directed graph with a "source" vertex $s \in V$. A source vertex is one in which there are no incoming edges: that is, the in-neighborhood is empty, $N^-(s) = \emptyset$. Show that one of the following two statements must be true:

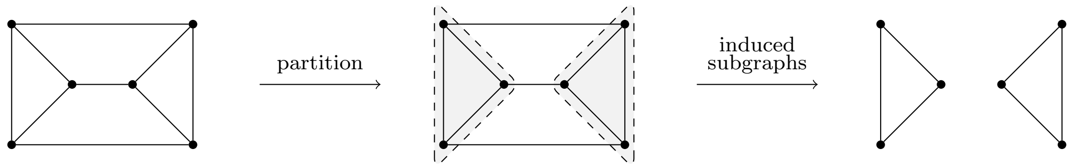
   - $\vec{G}$ has a cycle; that is, there are vertices $v_1, \ldots, v_m \in V$ such that

     $$(v_1, v_2), (v_2, v_3), \ldots, (v_{m-1}, v_m), (v_m, v_1) \in \vec{E}$$

   - $\vec{G}$ has a sink; that is, there is a vertex $t \in V$ with no outgoing edges: $N^+(t) = \emptyset$.

3. **Amoeba graphs split into isomorphic parts**

We say that a finite undirected graph $G = (V, E)$ on an even number of vertices is an *amoeba graph* if the set of vertices V can be partitioned into two disjoint subsets V_1 and V_2 such that the induced subgraphs $G[V_1]$ and $G[V_2]$ are isomorphic. For example, the complete graph on $2n$ vertices is an amoeba graph because the induced subgraph on any subset of n vertices is the complete graph on n vertices. Another 6-vertex example is given below:



Show that for every $n \geq 2$, there is a simple undirected graph on $2n$ vertices that is *not* an amoeba graph.

4. **Bipartite Handshaking Lemma**

Let $G = (V, E)$ be a finite undirected bipartite graph with bipartition (V_1, V_2). Using the fact that every edge $e \in E$ has one endpoint in V_1 and one endpoint in V_2, derive a specialization of the Handshaking Lemma to bipartite graphs that relates the degrees of the vertices in V_1 to the degrees of vertices in V_2. Suppose that every vertex in V_1 has degree r_1 and every vertex in V_2 has degree r_2. Use your new handshaking lemma to compute $|V_1|/|V_2|$ in terms of the degrees r_1 and r_2.