Math 154 - Discrete Math and Graph Theory
Homework 4
Due **Tuesday, February 18th, 11:59pm**

Instructions: It is highly recommended (though not required) that you type your answers. It is your responsibility to make any handwriting clear and legible for grading. A LaTeX template for the homework is provided on Canvas. Please start each problem on a new page.

We will only be grading some of the problems below for correctness. However, because all of the concepts are important, we will not reveal which problems are being graded for correctness until after the assignment has been submitted. The remaining problems will be graded for completeness (i.e., does it look like there was a good-faith effort to solve the problem?).

Additional textbook questions for practice (not graded): 4.1(a,b), 4.4 (Assume $k \geq 2$)

Problems:

1. **Superblock decomposition**

   Recall that in a block decomposition, each block is a maximal connected subgraph with no cut vertex. Let's define a *k-superblock* to be a maximal k-connected subgraph of size at least k. Notice that $k = 2$ corresponds to our usual definition of a block.

   Let $k \geq 2$ and let $B_1, \ldots, B_m$ be all of the k-superblocks of some simple undirected graph G. Recall that the maximality of each block means that $B_i \not\subseteq B_j$ for any pair of blocks i and j.

   (a) Prove that $|B_i \cap B_j| \leq k - 1$ for all choices of $i, j \in \{1, \ldots, m\}$.

   (b) Let's construct a new graph $\mathcal{B}$ from G capturing the connectivity between the superblocks. We have two kinds of vertices:

   - Superblock vertices: A vertex b_i for each block B_i.
   - Block intersections: A vertex b_S for each subset of vertices $S \subseteq V(G)$ such that there exists B_i and B_j such that $|S| = k - 1$ and $B_i \cap B_j = S$.

   We add an edge between b_i and b_S if $B_i \cap S \neq \emptyset$.

   Prove/Disprove: If $\mathcal{B}$ is connected, then $\mathcal{B}$ is a tree.

2. **Can you ear me?**

   (a) Let G be any simple undirected graph with an ear decomposition. Prove that G has no cut vertex.

   (b) Let G be any simple undirected connected graph on n vertices. What is the maximum number of edges you might need to add to G so that it has an ear decomposition? Prove that your answer is tight—that is, there is some graph G requires adding that many edges, and all graphs admit an ear decomposition after adding that many edges.

3. **Large vertex cuts imply large cycles**

Let $k \geq 2$. Our goal for this problem is to show that any k-connected simple undirected graph G with at least $2k$ vertices contains a cycle of length at least $2k$.

Let C be the set of vertices contained in some cycle of maximum length in G. We want to show that $|C| \geq 2k$. Let's start by proving something a bit weaker than what we ultimately want to show.

(a) Prove that $|C| \geq k+1$. *Hint: Consider the degrees of vertices in G.*

If $|C| \geq 2k$, then there is nothing left to show. So, let's assume by contradiction that $|C| < 2k$. Let $\overline{C} = V(G) - C$ be the set of vertices in G that are *not* part of the cycle.

(b) Show there are at least k vertices in C with an edge to a vertex in $\overline{C}$.

(c) Show there are at least 2 *adjacent* vertices in C with an edge to a vertex in $\overline{C}$. *Hint: Use the fact that $|C| < 2k$.*

Let $u, v \in C$ be the two adjacent vertices of C. Let $\overline{u}$ and $\overline{v}$ be the vertices they are adjacent to in $\overline{C}$. That is, $\{u, \overline{u}\} \in E(G)$ and $\{v, \overline{v}\} \in E(G)$. We will now show that $\overline{u}$ and $\overline{v}$ can be used to extend our cycle, contradicting the fact that it was maximal. There are two cases:

(d) Case 1: Suppose that $\overline{u} = \overline{v}$ or that $\overline{u}$ is connected to $\overline{v}$ in $G[\overline{C}]$ (the induced subgraph of the vertices in $\overline{C}$). Show that there is a cycle in G which is longer than $|C|$.

(e) Case 2: Suppose there is no path between $\overline{u}$ and $\overline{v}$ in $G[\overline{C}]$. Show that there is a cycle in G which is longer than $|C|$. *Hint: Use Menger's theorem and logic similar that required for part (c).*