Math 154 - Discrete Math and Graph Theory
Homework 6
Due **Monday, March 10th, 11:59pm**

Instructions: It is highly recommended (though not required) that you type your answers. It is your responsibility to make any handwriting clear and legible for grading. A LaTeX template for the homework is provided on Canvas. Please start each problem on a new page.

We will only be grading some of the problems below for correctness. However, because all of the concepts are important, we will not reveal which problems are being graded for correctness until after the assignment has been submitted. The remaining problems will be graded for completeness (i.e., does it look like there was a good-faith effort to solve the problem?).

Additional textbook questions for practice (not graded): 7.4(a,b), 7.5

Problems:

1. **Constraining colors using chromatic number**

   Let $c\colon V(G) \to \{1, \dots, k\}$ be a proper k-coloring of a simple graph G with chromatic number equal to k. Show that there is a vertex $v \in V(G)$ whose neighbors in G are colored with every other color. That is, $\{c(v)\} \cup \{c(u) : u \in N_G(v)\} = \{1, \dots, k\}$.

2. **Coloring regular graphs**

   Let G be a k-regular graph for some $k \geq 1$. That is, $d_G(v) = k$ for all $v \in V(G)$.

   (a) Suppose that G has a proper k-edge-coloring. Prove that any set of edges sharing the same color is a perfect matching.

   (b) Show that the edge chromatic number of G is $k + 1$ whenever G has a cut vertex. *Hint: Consider the connected components on either side of the cut vertex and use part (a).*

3. **Counting colorings**

   For any graph G, let $P_G(k)$ be the number of proper k-colorings of G.

   (a) Show that $P_G(k)$ satisfies the following recursive formula for any edge $e \in E(G)$:

   $$P_G(k) = P_{G \backslash e}(k) - P_{G/e}(k).$$

   Here, we are using $G \backslash e$ to denote the graph G with the edge e removed. Furthermore, we use G/e to denote the contraction of e in the graph G (Recall the graph contraction operation for a vertex set in Chapter 1.7).

   $P_G(k)$ is called the *chromatic polynomial* of G. Specifically, it is a polynomial in k because it can be written as

   $$P_G(k) = \alpha_0 + \alpha_1 k + \alpha_2 k^2 + \cdots + \alpha_d k^d$$

   for some coefficients $\alpha_i \in \mathbb{R}$ that can depend on properties of the graph such as the number of vertices or edges, but not k itself. Here, we assume that α_d is the largest coefficient of the polynomial not equal to 0. That is, the polynomial has degree d.

(b) Suppose G is a graph with n vertices, but no edges. What is $P_G(k)$?

(c) Suppose G is a graph with n vertices and exactly one edge. What is $P_G(k)$?

(d) Suppose G is a graph with n vertices with exactly two edges that share an endpoint. What is $P_G(k)$?

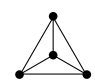
(e) Let G be any simple graph on n vertices. Show that the degree of the chromatic polynomial is n. What are the coefficients of k^n and k^{n-1}?

Hint: Look at all your previous examples and come up with a guess for the coefficients of k^n and k^{n-1} for general graphs. Finish the proof using induction and the recursive formula for $P_G(k)$.

4. **Are these graphs planar?**

For each list of numbers below, determine if there is a (simple, undirected) planar graph with that list as its degree sequence. If the answer is "yes", draw the planar embedding of the graph. If the answer is "no", prove that no such simple graph has that degree sequence.

For example, the answer for the sequence $(3, 3, 3, 3)$ is "yes" because of the following platonic solid graph:



(a) $(5, 5, 5, 5, 5, 5)$

(b) $(3, 3, 3, 3, 3, 3, 3)$

(c) $(5, 5, 5, 5, 5, 5, 5, 5)$

(d) $(3, 3, 4, 4, 5, 5)$