The Complexity of Bipartite Gaussian Boson Sampling

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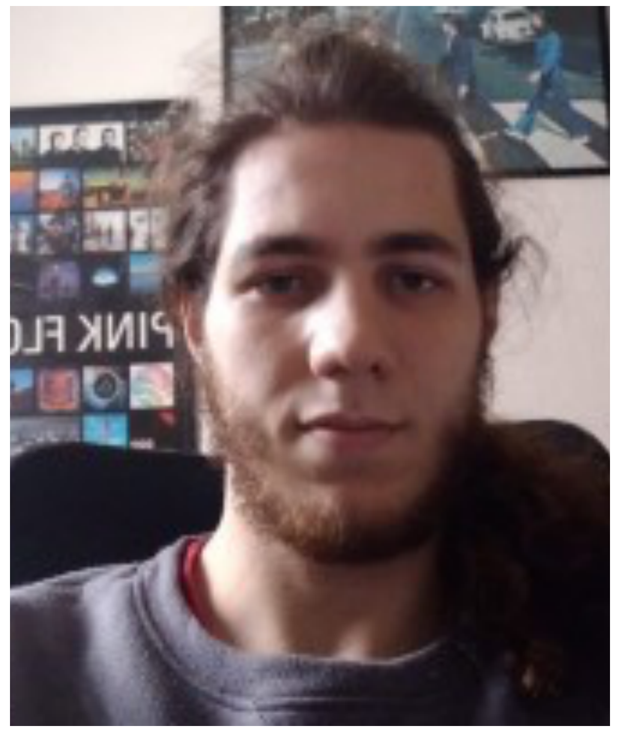

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Quantum computational advantage with linear optics

Is it hard to classically sample from the distributions produced by weak photonic quantum computers?

Strong candidates:

(Fock) BosonSampling [Aaronson, Arkhipov STOC 11]

Fermion Sampling with magic input states [Oszmaniec et al. QIP 22]

Gaussian Boson Sampling [Lund et al. PRL 14, Hamilton et al. PRL 17]

Scattershot BosonSampling

Problems:

1) Disconnected landscape of conjectures
2) Extra conjectures needed to accommodate experimental costs

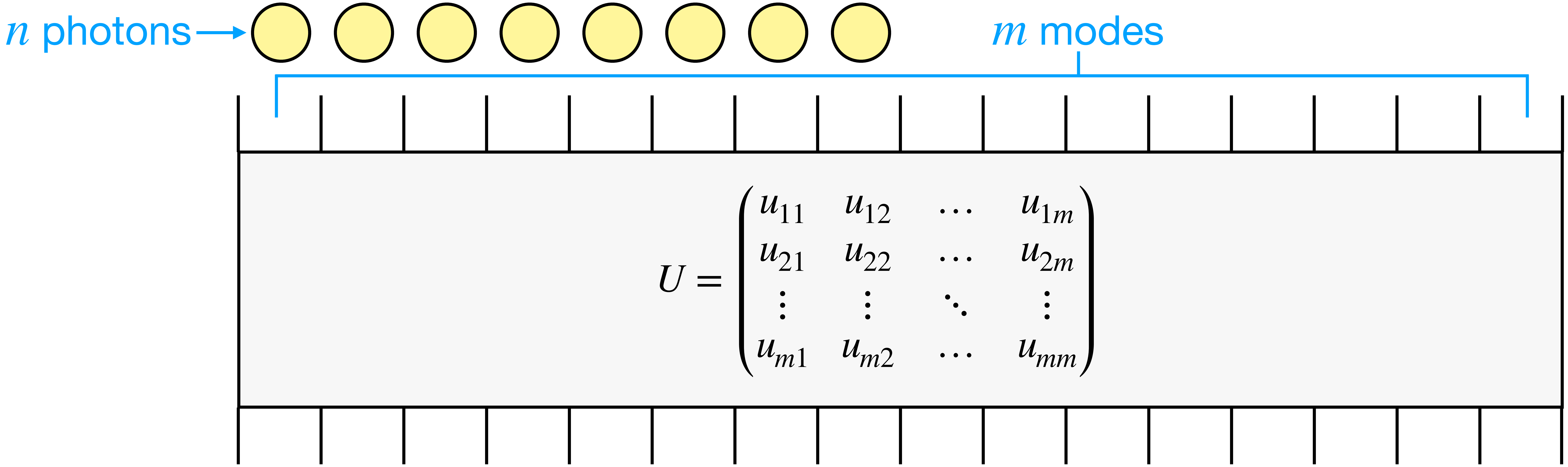
BipartiteGBS - quantum advantage with fewer assumptions

Bipartite Gaussian Boson Sampling (BipartiteGBS):
Method for programming a Gaussian Boson Sampling device

- Connects Gaussian Boson Sampling with (Fock) BosonSampling
- Removes a conjecture that is required for BosonSampling:
  Theorem: Hardness when modes are quadratic in the number of photons
- Versatile tool for building future hardness arguments:
  Theorem: Hardness with constantly-many collisions

BosonSampling revisited

Theorem [AA]: It is hard* to classically sample from the output of a BosonSampling experiment (even approximately).



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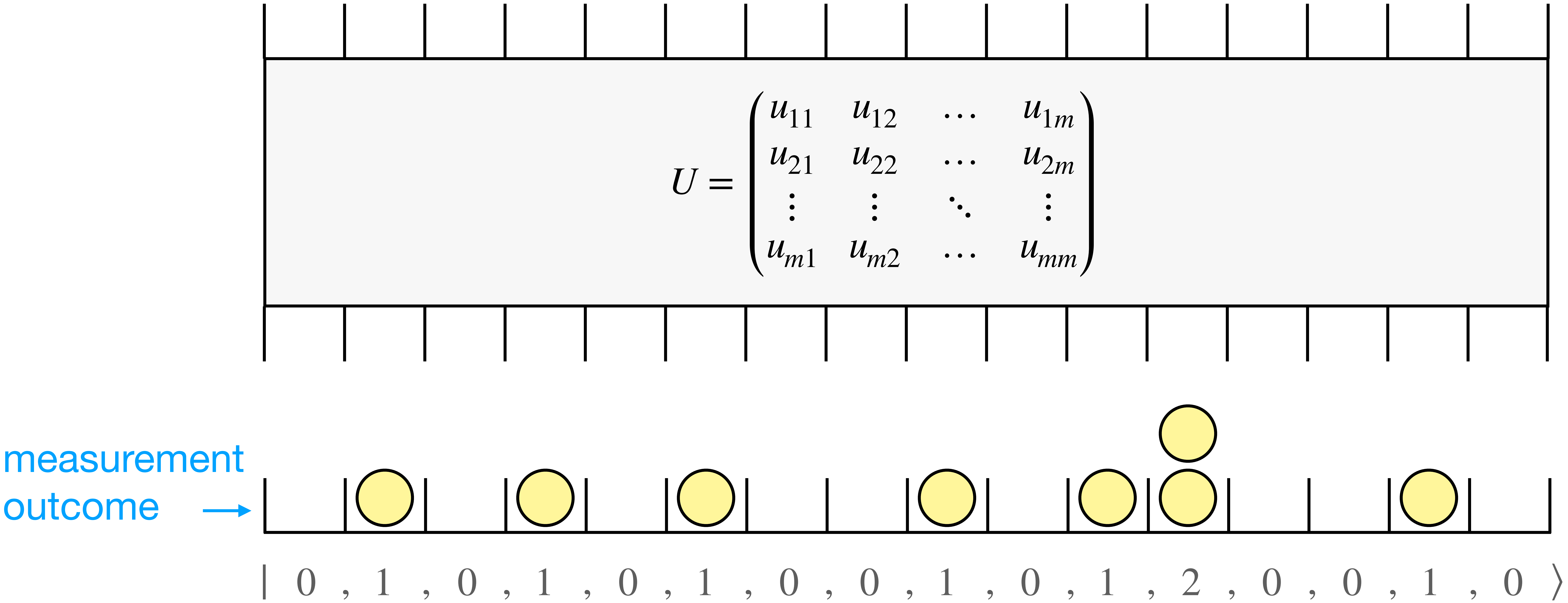
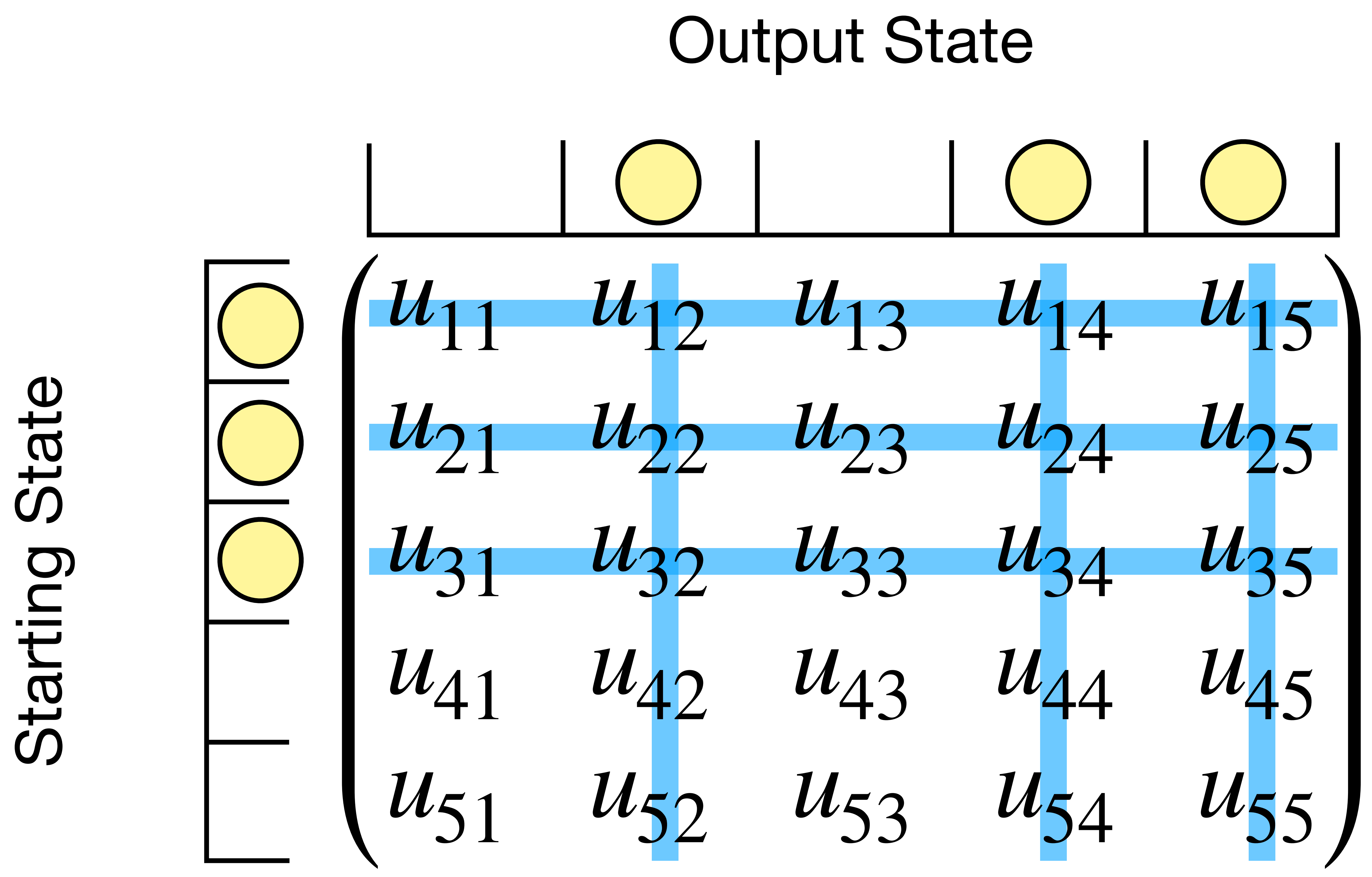
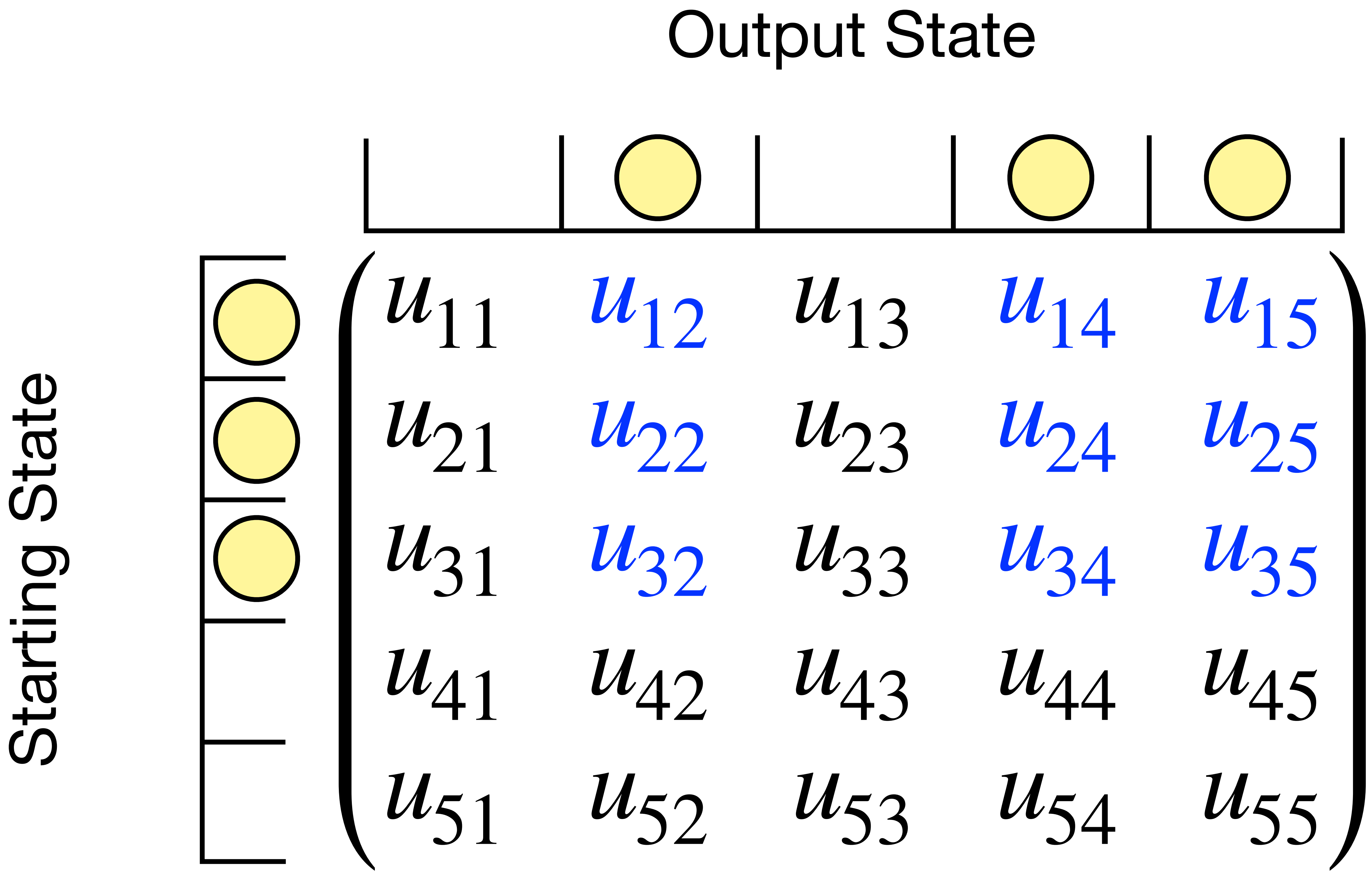


BosonSampling probabilities given by permanent



Pr[]

BosonSampling probabilities given by permanent



$$\begin{pmatrix} u_{11} & u_{12} & u_{13} & u_{14} & u_{15} \\ u_{21} & u_{22} & u_{23} & u_{24} & u_{25} \\ u_{31} & u_{32} & u_{33} & u_{34} & u_{35} \\ u_{41} & u_{42} & u_{43} & u_{44} & u_{45} \\ u_{51} & u_{52} & u_{53} & u_{54} & u_{55} \end{pmatrix}$$

Pr[] $= \text{Per} \circ \begin{pmatrix} u_{12} & u_{14} & u_{15} \\ u_{22} & u_{24} & u_{25} \\ u_{32} & u_{34} & u_{35} \end{pmatrix}$

Quantum computational advantage from linear optics

Theorem [AA]: There is no classical polynomial-time algorithm to approximately sample from the output distribution of a BosonSampling experiment.

Modulo four conjectures:

1) Non-collapse of the polynomial hierarchy
2) Gaussian permanent estimation is #P-hard
3) Anti-concentration of Gaussian permanents
4) The $n \times n$ submatrices of an $n^2 \times n^2$ unitary matrix look Gaussian

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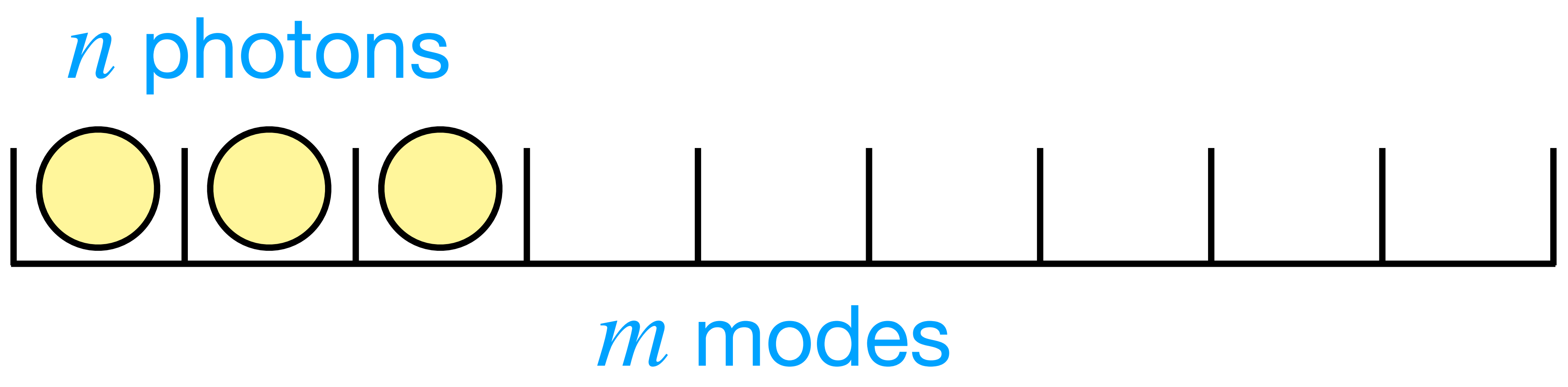
Physical Interpretation:
Sufficient to have quadratically more modes than photons ($m \approx n^2$).



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Theorem [AA]: Can remove conjecture if $m \approx n^5$

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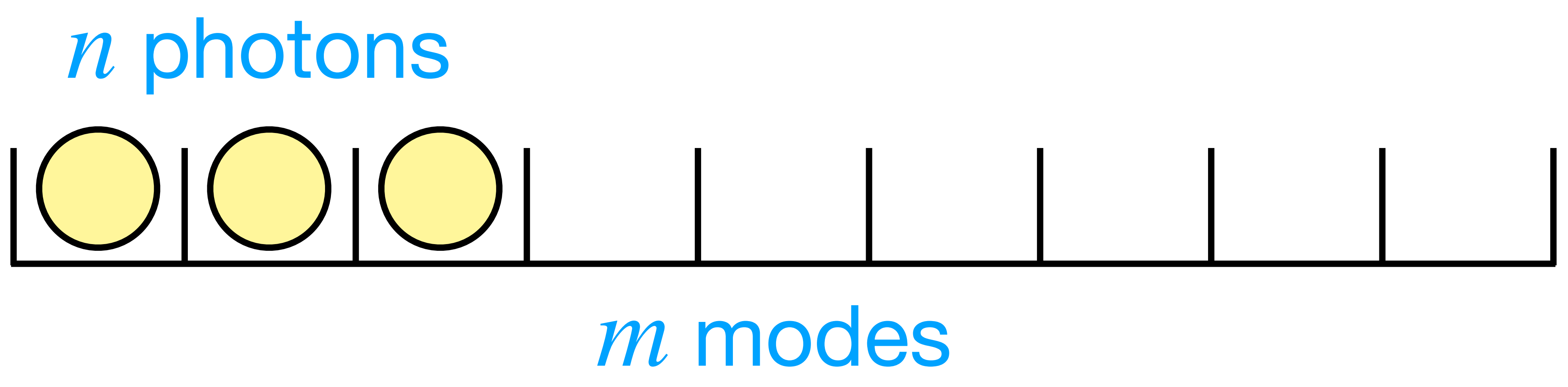
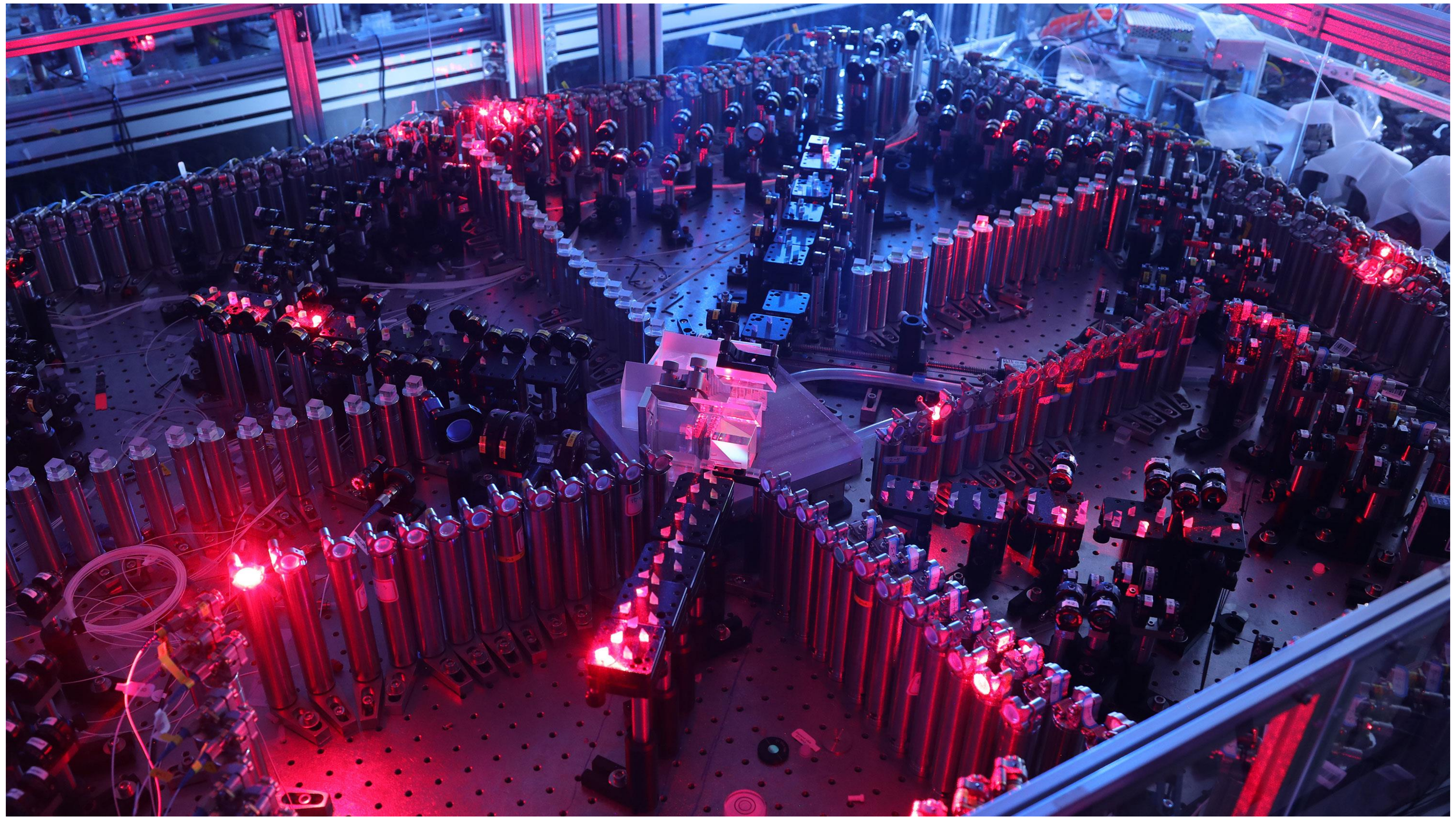


Experimental Gaussian Boson Sampling uses few modes

Gaussian Boson Sampling Experiments
Two recent experiments claiming quantum advantage [Science 20, PRL 21]

Experiment #1: 45 photons, 100 modes

Experiment #2: 113 photons, 144 modes



Credit: Quantum computational advantage using photons [Zhong, et al. Science 20]

Hardness of classical sampling with few modes

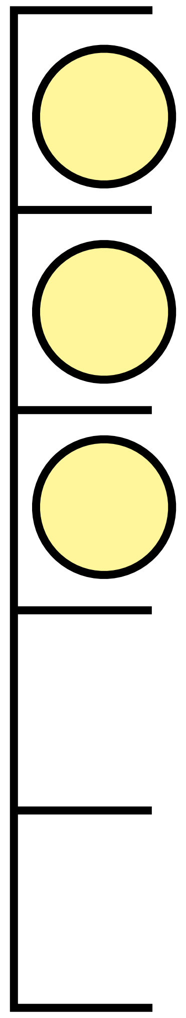
Theorem: There is no classical polynomial-time algorithm to approximately sample from the output distribution of a BipartiteGBS experiment whenever $m \approx \mathbb{E}[n]^2$.

→ Requires only three of the four BosonSampling conjectures

BosonSampling conjectures:

1) Non-collapse of the polynomial hierarchy
2) Gaussian permanent estimation is #P-hard
3) Anti-concentration of Gaussian permanents
4) ~~The $n \times n$ submatrices of an $n^2 \times n^2$ unitary matrix look Gaussian~~

Submatrices of Haar random unitaries



$$\begin{pmatrix} u_{11} & \color{blue}{u_{12}} & u_{13} & \color{blue}{u_{14}} & \color{blue}{u_{15}} \\ u_{21} & \color{blue}{u_{22}} & u_{23} & \color{blue}{u_{24}} & \color{blue}{u_{25}} \\ u_{31} & \color{blue}{u_{32}} & u_{33} & \color{blue}{u_{34}} & \color{blue}{u_{35}} \\ u_{41} & u_{42} & u_{43} & u_{44} & u_{45} \\ u_{51} & u_{52} & u_{53} & u_{54} & u_{55} \end{pmatrix}$$

$\Pr[$  $] = \mathrm{Per} \circ \begin{pmatrix} u_{12} & u_{14} & u_{15} \\ u_{22} & u_{24} & u_{25} \\ u_{32} & u_{34} & u_{35} \end{pmatrix}$

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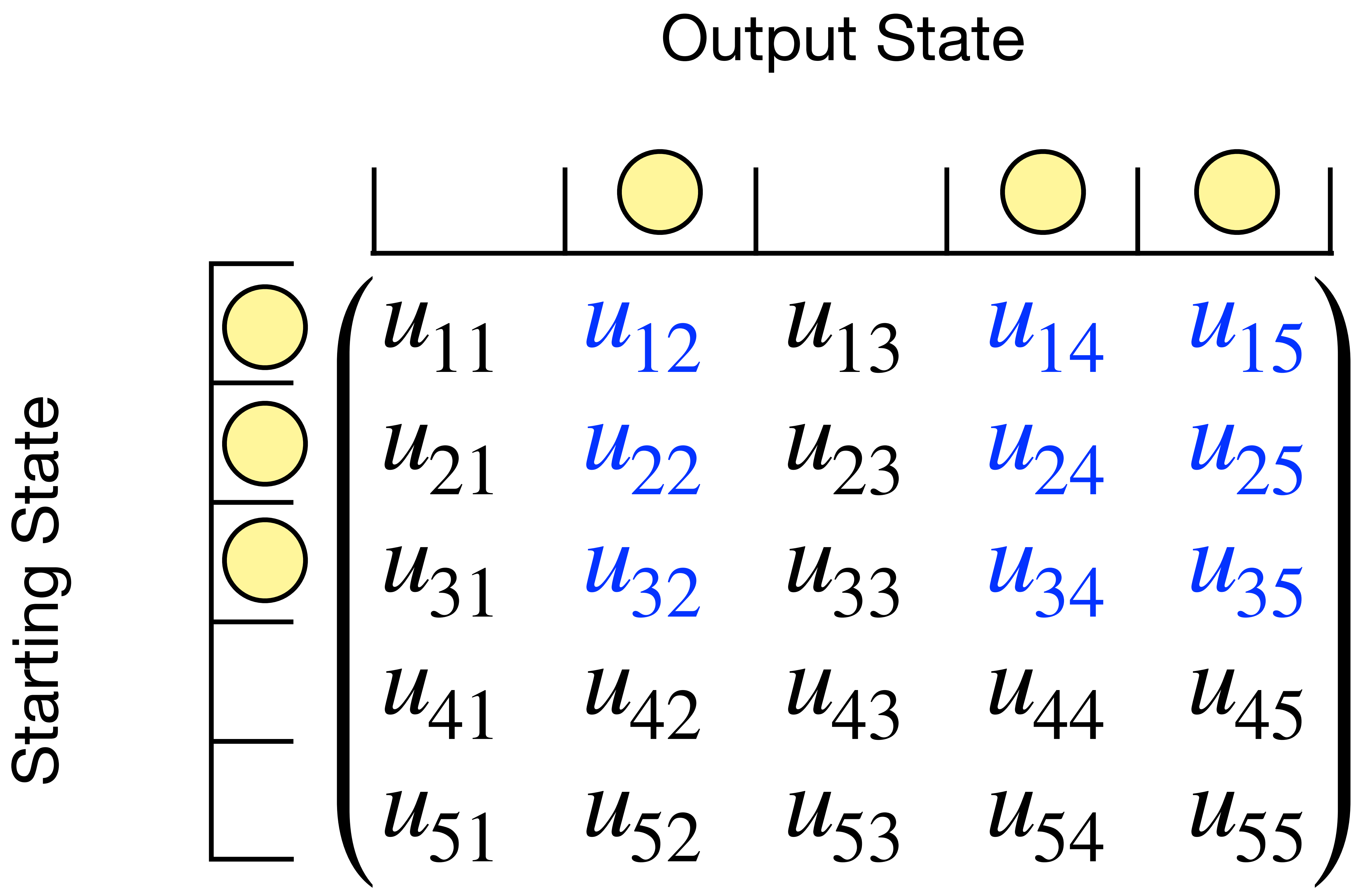
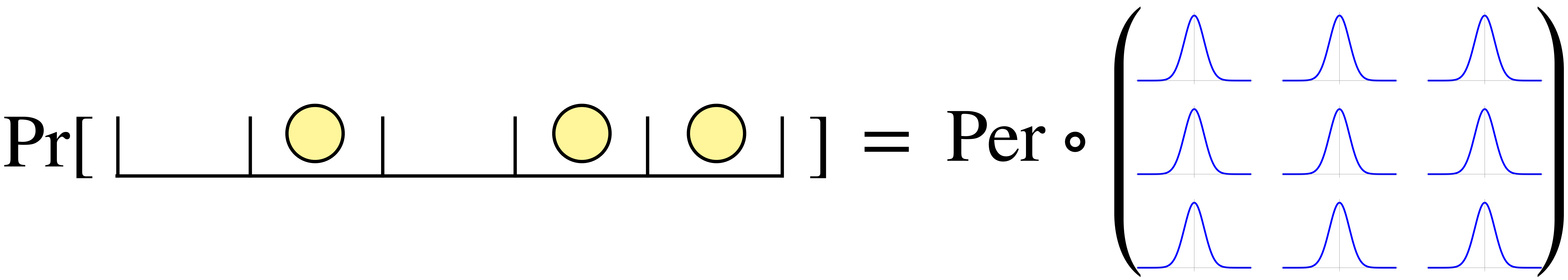


Problem: Only rigorous convergence bounds whenever $m = \omega(n^5)$.

Required Property: this submatrix approximates a matrix with i.i.d. complex Gaussian entries whenever the unitary is Haar random.



Prior work on submatrices of Haar random unitaries

Theorem [Jiang 2006]**:** The $n \times n$ submatrices of random $m \times m$ real orthogonal matrices converge (in total variation) to real Gaussian matrices whenever $m = \omega(n^2)$.

→ *Issues for BosonSampling:*

1) Real matrices rather than complex ones

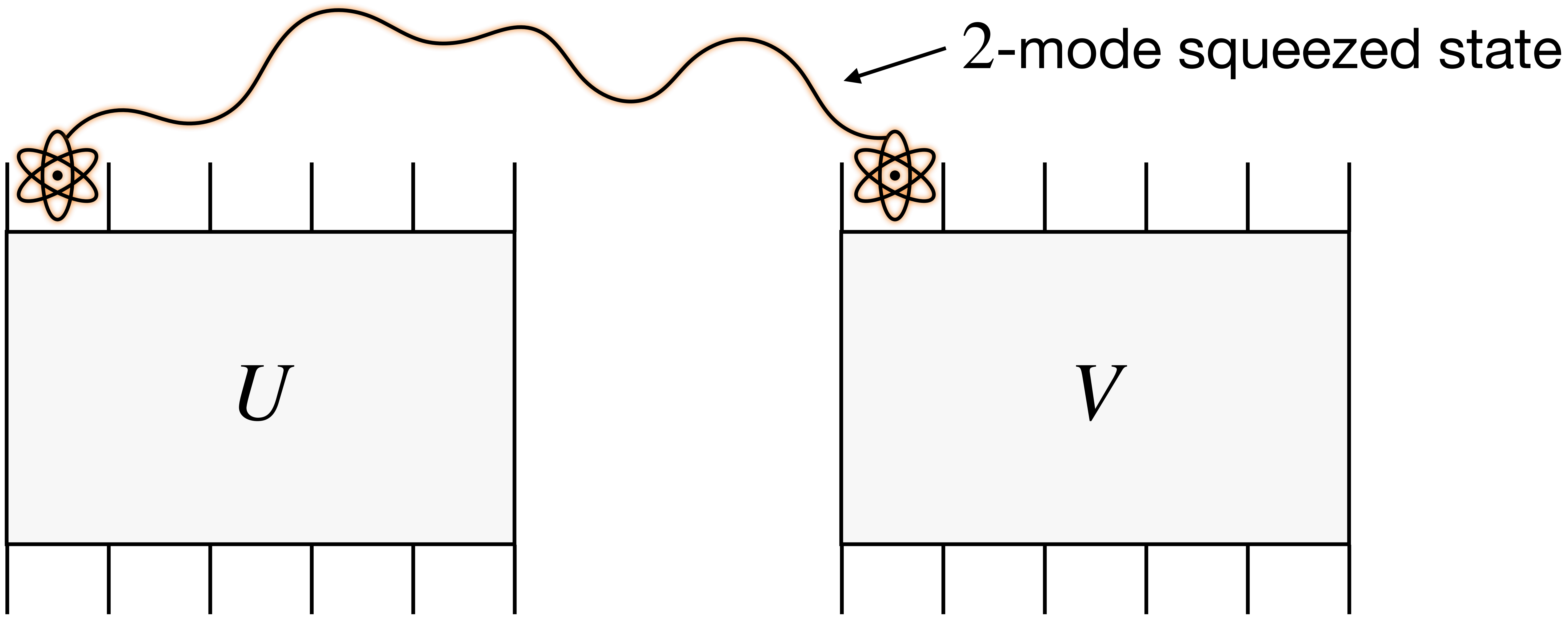
2) Does not bound the *rate* of this convergence

Theorem [AA]**:** Variation distance is $O(\delta)$ whenever $m \geq \frac{n^5}{\delta} \log^2 \frac{n}{\delta}$.

→ We do not try to improve this theorem directly!

Avoid conjecture by directly encoding Gaussian entries

BipartiteGBS: There is a Gaussian Boson Sampling experiment such that the output probabilities are governed by the permanents of submatrices of *arbitrary* matrices.



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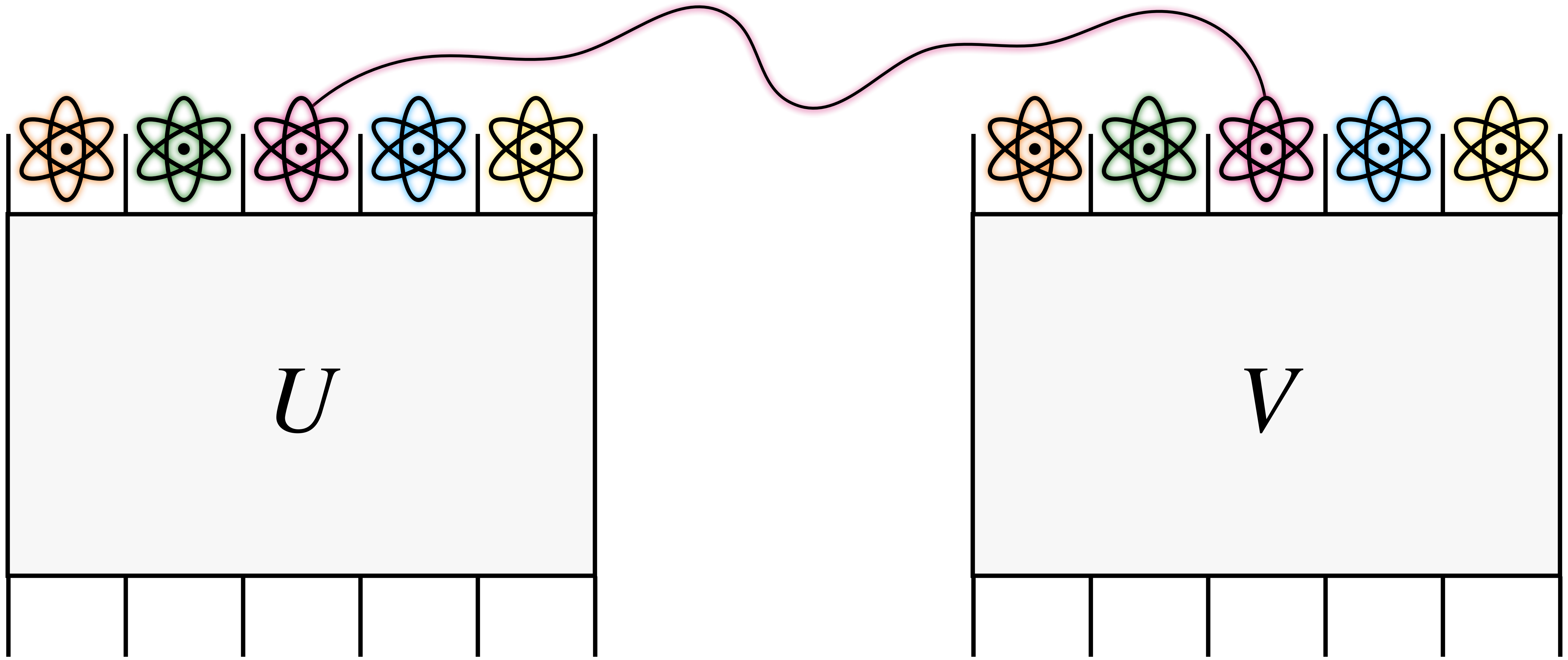
Different squeezing parameters



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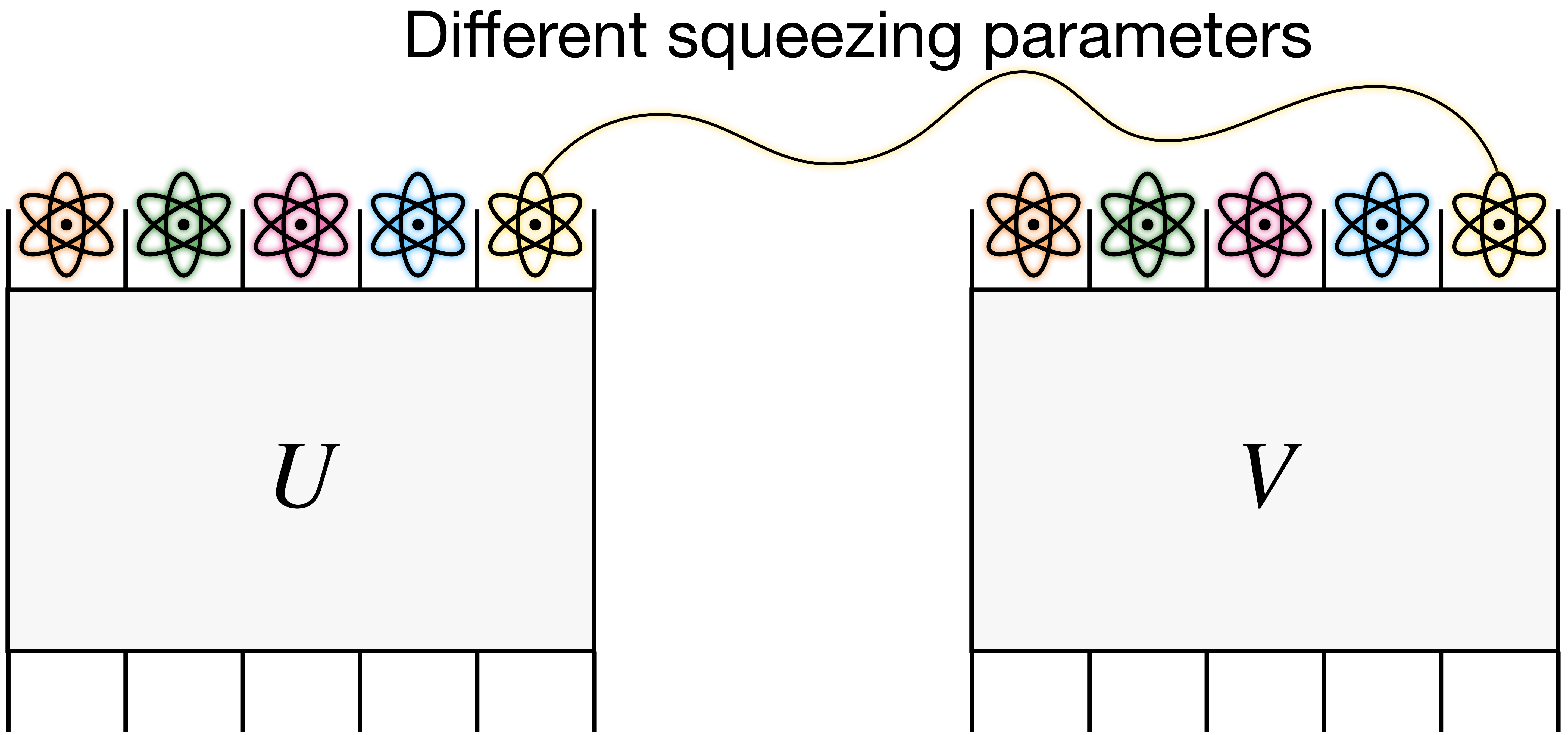

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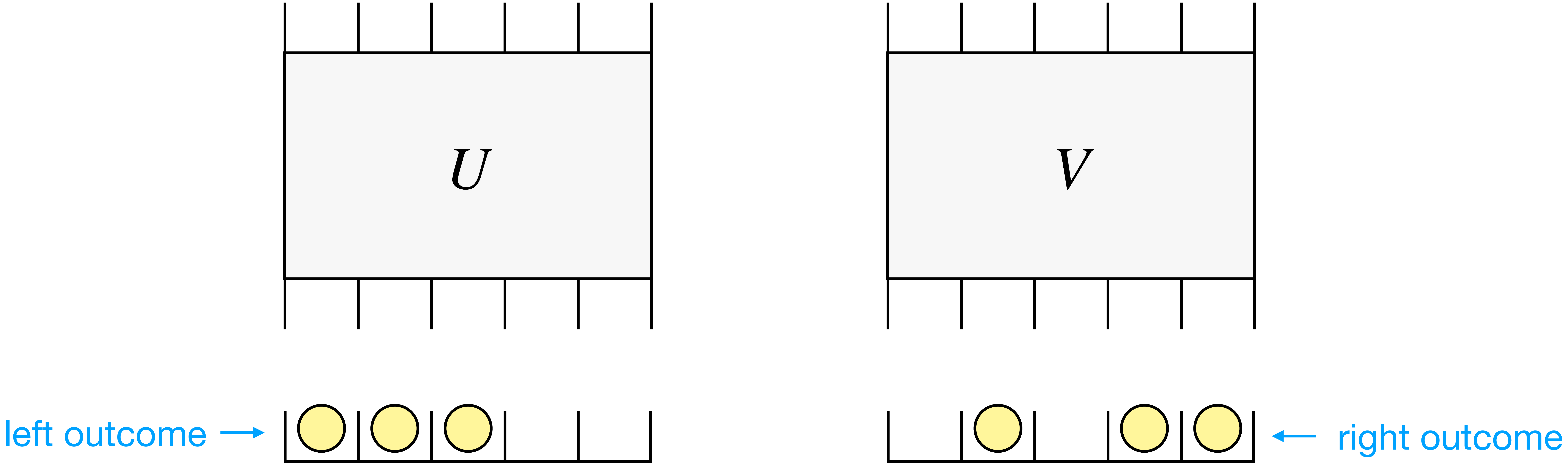

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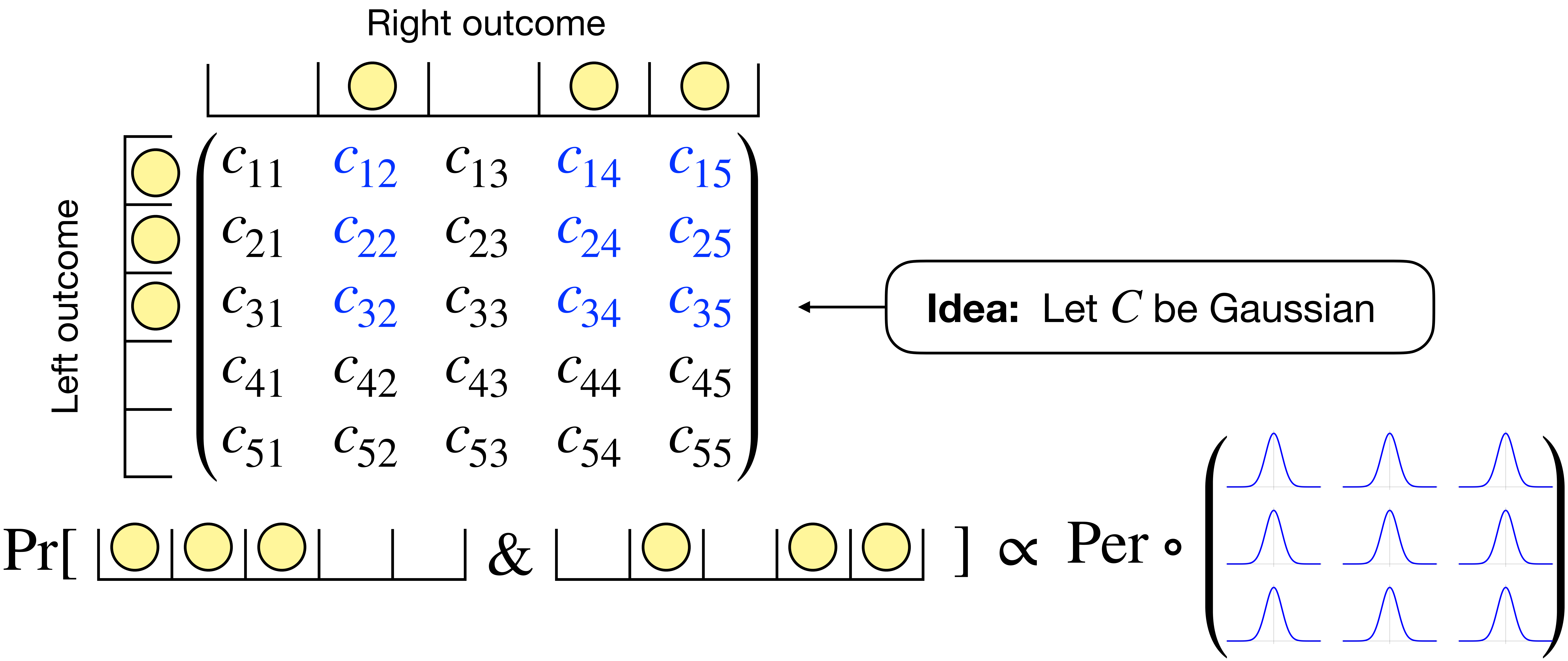
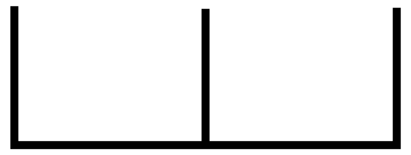
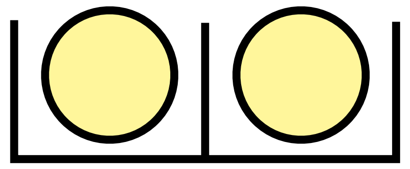
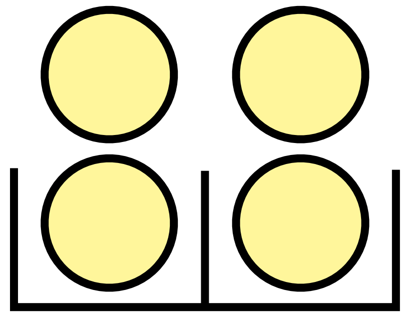


BipartiteGBS - input states and output probabilities

Input states:

$$S_2(r)\,|\,0{,}0\rangle = \frac{1}{\cosh r}\sum_{n=0}^{\infty}(\tanh r)^n\,|\,n,n\rangle$$

Expected photon number
$\langle n\rangle = \sinh^2(r)$

$S_2(1)\,|\,0{,}0\rangle =$ .65  + .49  + .38  + $\cdots$

Output probabilities:

Left modes: $S = |\,s_1, \ldots, s_m\rangle$

Right modes: $T = |\,t_1, \ldots, t_m\rangle$

Singular value decomposition of arbitrary matrix with singular values in $[0,1)$.

$$C = U\,\mathrm{diag}(\tanh r_i)\,V^T$$

$$\Pr[S\;\&\;T] = \frac{1}{\mathcal{Z}}\frac{|\,\mathrm{Per}(C_{S,T})\,|^2}{\prod_{i=1}^{m} s_i!t_i!}$$

$$\mathcal{Z} = \prod_{i=1}^{m}\cosh^2(r_i)$$

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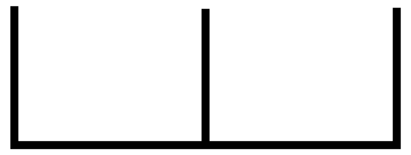
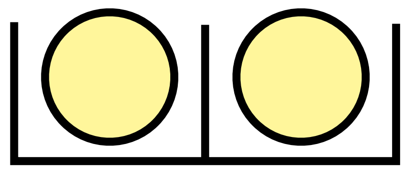
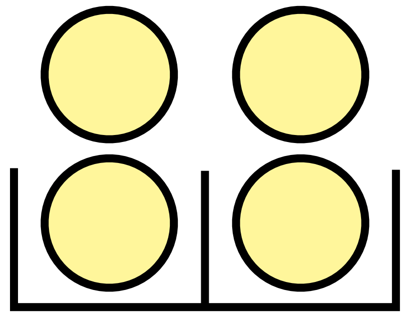
$S_2(1)\,|0,0\rangle = $ .65  + .49  + .38  + $\cdots$

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Scattershot BosonSampling:
$V = I$ and r_i's constant

$$C = U\,\mathrm{diag}(\tanh r_i)\,V^T$$

$$\mathcal{Z} = \prod_{i=1}^{m}\cosh^2(r_i)$$

Proof outline for main theorem

1) Suppose there is a classical algorithm that samples from the output distribution of a BipartiteGBS experiment

$$\|\mathcal{D}_C - \mathcal{D}'_C\| \leq \beta$$

Gaussian Permanent Estimation: Given $n \times n$ matrix X with i.i.d. standard complex Gaussian entries, estimate $|\text{Per}\, X|^2$ to $(\epsilon n!)$-additive accuracy with probability $1 - \delta$.

→ *Conjecture* [AA]: Gaussian Permanent Estimation is #P-hard.

2) Hide the $n \times n$ Gaussian matrix X in an $m \times m$ Gaussian matrix C

3) BipartiteGBS with matrix C has output with probability proportional to $|\text{Per}\, X|^2$

4) Estimate this probability using Stockmeyer counting on $\mathcal{D}'_C$ to compute $|\text{Per}\, X|^2$

5) By conjecture, algorithm for $|\text{Per}\, \text{X}|^2$ implies collapse of polynomial hierarchy

Is the Stockmeyer counting argument good enough?

Gaussian Permanent Estimation: Estimate $|\mathrm{Per}\,X|^2$ to $(\epsilon n!)$-additive accuracy

Lemma: There is a $\mathsf{BPP}^{\mathsf{NP}}$ algorithm that estimates $|\mathrm{Per}\,X|^2$ to additive error

$$\epsilon\left(\mathcal{Z} m^{(3/2)n}\binom{m}{n}^{-2}\right)$$

- Accuracy of this estimate depends on $\mathcal{Z} = \prod_{i=1}^{m}\cosh^2(r_i)$
- $\mathcal{Z}$ depends on the singular values of matrix with i.i.d. Gaussian entries

Have we just traded one problem in random matrix theory with another?

Is the Stockmeyer counting argument good enough?

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- Accuracy of this estimate depends on $\mathscr{Z} = \prod_{i=1}^{m} \cosh^2(r_i)$
- $\mathscr{Z}$ depends on the singular values of matrix with i.i.d. Gaussian entries

Theorem: $\Pr[\mathscr{Z} \leq \frac{1}{\delta} e^{\sqrt{m}}] \leq \delta$ whenever $m = \Theta(n^2)$.

- *Proof tool:* for Gaussian C, we have $\mathscr{Z}^{-1} = \det(I - CC^\dagger)$.

Summary and future directions

Theorem: Hard to approximately sample from the output distribution of a GBS experiment in the no-collision regime ($m \approx n^2$) under BosonSampling conjectures.

Can we get hardness in the high-collision regime ($m = o(n^2)$)?

→ Probabilities given by permanents of Gaussian matrices with repeated rows/columns

Repeated Gaussian Permanent Estimation: Given $c \times c$ Gaussian matrix X and collision patterns $S = (s_1, \ldots, s_c)$, $T = (t_1, \ldots, t_c)$ with $s_1 + \ldots + s_c = t_1 + \ldots + t_c = n$, estimate $|\mathrm{Per}\, X_{S,T}|^2$ to $(\epsilon n! s_1! \cdots s_c! t_1! \cdots t_c!)$-additive accuracy with probability $1 - \delta$.

→ *Speculative conjecture:* Repeated Gaussian Permanent Estimation is #P-hard